COLLEGE PHYSICS - II

Vector Calculus and Analog Electronics

S.Y. B.Sc.

Semester - III

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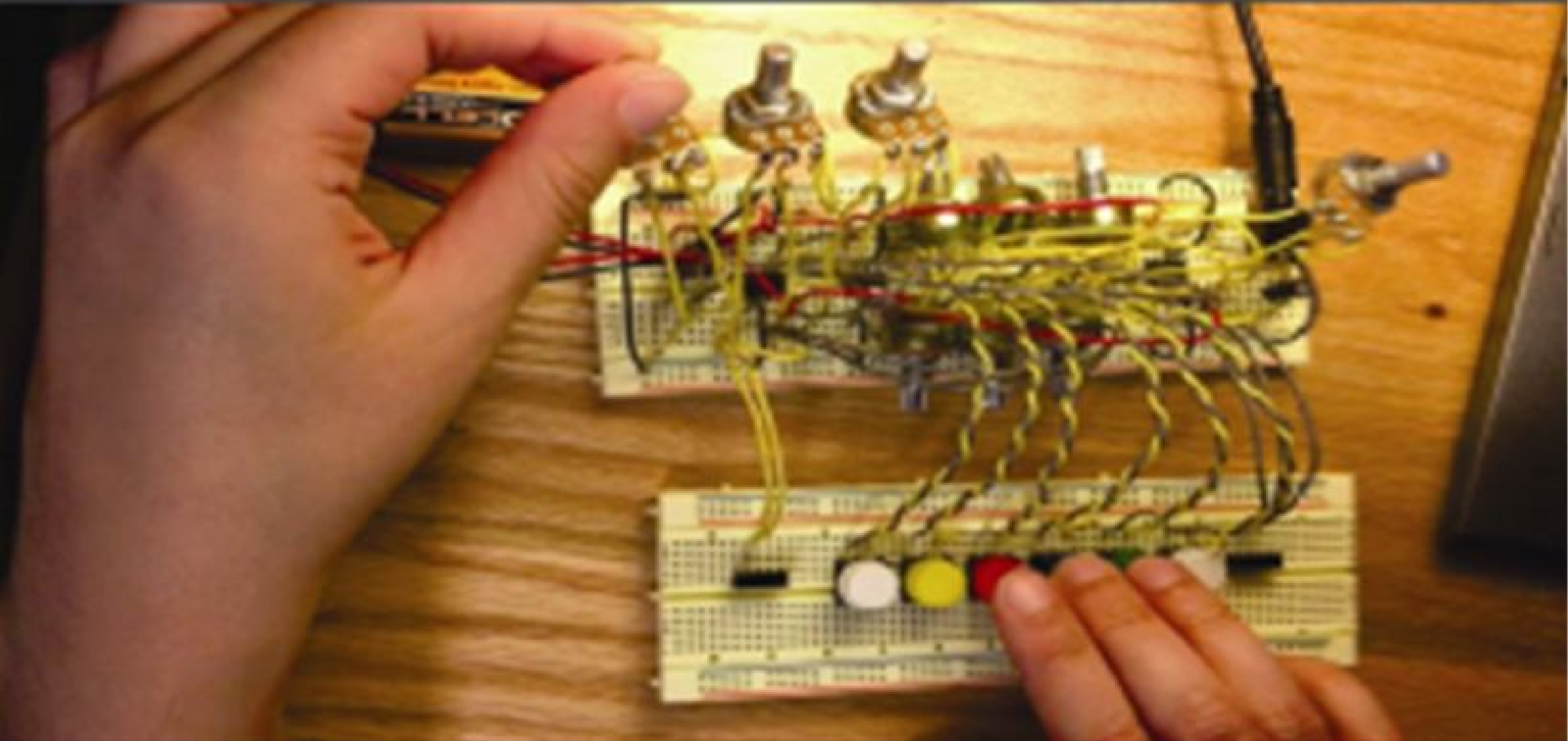

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PHYSICS - II

Vector Calculus and Analog Electronics

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PREFACE

We are pleased to present this book **"College Physics - II"** on **"Vector Calculus and Analog Electronics."** This book is prepared according to new syllabus prescribed by the University of Mumbai for S. Y. B.Sc. - Semester III in Physics course.

We, the authors were free to choose the methods and style of presentation. The explanation of the subject matter has been coupled with variety of solved examples which have been introduced at appropriate stages.

We hope that this will be extremely useful to the students in understanding the subject. We welcome the suggestion for improvement.

We thanks the Director, Shri. K. N. Pandey and the team of Himalaya Publishing House Pvt. Ltd. for bringing out this book.

Mumbai **AUTHORS**

SYLLABUS

USPH302: Vector Calculus and Analog Electronics

Learning Outcomes:

On successful completion of this course students will be able to:

(i) Understand the basic concepts of mathematical physics and their applications in physical situations.

(ii) Understand the basic laws of electrodynamics and be able to perform calculations using them.

(iii) Understand the basics of transistor biasing, operational amplifiers, their applications.

(iv) Understand the basic concepts of oscillators and be able to perform calculations using them.

(v) Demonstrate quantitative problem solving skill in all the topics covered.

Unit I: Vector Calculus **15 Lectures**

(1) Line, Surface and Volume Integrals, The Fundamental Theorem of Calculus, The Fundamental Theorem of Gradient, The Fundamental Theorem of Divergence, The Fundamental Theorem of Curl (Statement and Geometrical Interpretation is included, Proof of these theorems are omitted). Problems Based on these Theorems are Required to be Done.

(ii) Curvilinear Coordinates: Cylindrical Coordinates, Spherical Coordinates.

Unit II: Analog Electronics **15 Lectures**

(i) Transistor Biasing, Inherent Variations of Transistor Parameters, Stabilisation, Essentials of a Transistor Biasing Circuit, Stability Factor, Methods of Transistor Biasing, Base Resistor Method, Emitter Bias Circuit, Circuit Analysis of Emitter Bias, Biasing with Collector Feedback Resistor, Voltage Divider Bias Method, Stability Factor for Potential Divider Bias.

(ii) General Amplifier Characteristics: Concept of Amplification, Amplifier Notations, Current Gain, Voltage Gain, Power Gain, Input Resistance, Output Resistance, General Theory of Feedback, Reasons for Negative Feedback, Loop Gain.

(iii) Practical Circuit of Transistor Amplifier, Phase Reversal, Frequency Response, Decibel Gain and Band Width.

Unit III: Analog Electronics **15 Lectures**

(i) Oscillators: Introduction, Effect of Positive Feedback. Requirements for Oscillations, Phase Shift Oscillator, Wien Bridge Oscillator, Colpitt's Oscillator, Hartley Oscillator.

(ii) Operational Amplifiers: Introduction, Schematic Symbol of OPAMP, Output Voltage from OPAMP, AC Analysis, Bandwidth of an OPAMP, Slew Rate, Frequency Response of an OPAMP, OPAMP with Negative Feedback, Inverting Amplifier, Non-inverting Amplifier, Voltage Follower, Summing Amplifier, Applications of Summing Amplifier, OPAMP Integrator and Differentiator, Critical Frequency of Integrator, Comparator.

CONTENTS

UNIT I

Chapter 1

VECTOR CALCULUS

1.1 Introduction

The physical meaning of the divergence and curl of a vector function is understood in terms of certain integrals and equations relating such integrals. In this chapter we shall focus on the theory of vector integration over vector fields. Also, we shall define line integral, surface integral and volume integral and some important applications. In this, we will learn, a line integral is a natural generalisation of a definite integral and a surface integral is generalisation of double integral.

The integrand may be either a vector or a scalar function, such as scalar line integral of a vector, the scalar surface integral of a vector and a volume integrals of both scalars and vectors.

The corresponding formulae serve as powerful tools in many applications.

1.2 Vector Integration

There are three kinds of integrals; line, surface and volume integrals which depends on the nature of problem. The integral can be scalar or a vector field giving rise to various possibilities. The most interesting may be; the line integral of a vector, the surface integral of a vector and volume integrals of both scalars and vectors. Let us begin with the concept and later to acquaint with their applications.

1.2.1 Line Integrals

The line integral is the integration of a vector function along a curve. The function to be integrated may be a scalar field or a vector field.

Let $\vec{V}$ be a continuous and single valued vector function in the region *R*. The line integral of $\vec{V}$ is written as,

$$\int_P^Q \vec{V} \cdot \vec{dl}$$

Simply $\int_C \vec{V} \cdot \vec{dl}$

where, *C* is the curve along which the integration is performed.

$\vec{dl}$ is an infinitesional vector displacement along the curve *C*. *P* and *Q* are initial and final points on the curve.

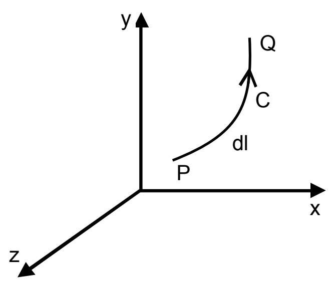


Fig. 1.1: Open Regular Curve and Contour C

Thus, line integral depends on the vector function $\vec{V}$, curve *C* joining two points *P* and *Q*. The curve *C* is called as "path of integration".

Note: A curve which consists of a finite number of regular arcs joined end to end, and which does not interest itself is called **regular curve**.

A regular arc is represented as,

$\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$

Here s is length of arc.

Since $\vec{V} \cdot \vec{dl}$ is a scalar, the line integral is also scalar. The line integral is also known as **curvilinear integral** or **path integral**.

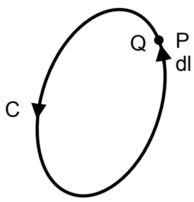


Fig. 1.2: Closed Regular Curve

The line integral ever a closed path C [i.e., points P and Q coincides] the symbol $\oint_c$ instead $\int_c$ is used. The line integral around a closed path or a curve may or may not be zero.

The vector fields whose line integrals vanishesh are called conservative, such vector fields are of considerable importance in the study of electromagnetism and mechanics.

$$\therefore \oint_c V \cdot dl = 0$$

Here line integral depends only on V and on the initial point P and end point Q but not on the path, i.e., curve from P to Q.

In physics we often come across line integrals of the type (i) $\int_c \phi\, dl$; (ii) $\int_c V\, dl$ and $\int_c V \times dl$, where, $\phi = \phi(x, y, z)$ is a scalar point function representing a scalar field $\vec{V} = \vec{V}(x, y, z)$ is a vector point function representing a vector field, [C is some countour, i.e., path of integration]

In component form we may write,

$$\int_c \vec{V} \cdot \vec{dl} = \int_c (V_x \hat{i} + V_y \hat{j} + V_z \hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (V_x\, dx + V_y\, dy + V_z\, dz)$$

This form is frequently used to evaluate line integrals.

If curve C is given by parametric equations $x = x(t)$, $y = y(t)$ and $z = z(t)$ then the line integral becomes,

$$\int \vec{V} \cdot \vec{dl} = \int_{t_P}^{t_Q} \left[Vx(t)\frac{dx}{dt} + Vy(t)\frac{dy}{dt} + Vz(t)\frac{dz}{dt} \right] dt$$

Example: The work done by a force field $\vec{F}$ along a path C from point a to another point b is

$$W = \int_a^b \vec{F} \cdot \vec{dr}$$

If total work done by force field around any closed path varishes then,

$$W = \oint \vec{F} \cdot \vec{dr} = 0$$ and $\vec{F}$ is said to be conservative.

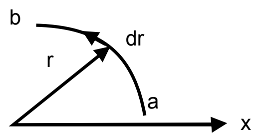


Fig. 1.3: Work: Path of integration

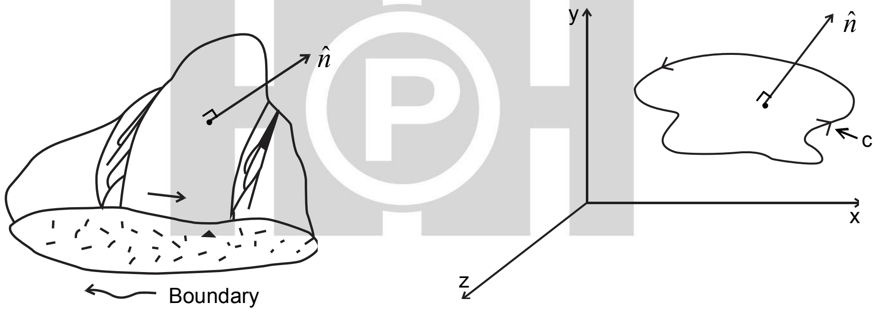
1.2.2 Surface Integrals

Surface integrals are the integration takes place over or along the surface rather than the curve. As seen in previous section line integral is represented as parametric function of the curve. Similarly, we shall define surface integrals in terms of a parametric representation for the surface.

Let S be a smooth surface bounded by a regular closed curve C. Let $\vec{V}$ be a single valued and continuous vector function defined over the region R of the surface, S. We subdivide the surface S into large number of small surface elements each of area da_i. For each element a point is chosen on it and the value of $\vec{V_i}$ at that point found. The scalar product of each elemental surface area $\overrightarrow{da_i}$ with corresponding value of $\vec{V_i}$ is found and sum of these computed.

Thus surface integral of a vector $\vec{V}$ over the surface S is defined as the integral of the components of $\vec{V}$ along the normal to the surface.

If $\vec{V}$ is a vector function, a surface integral of V over S is, $\int_S \vec{V} \cdot \hat{n}\, da$, where, S is the surface over which the integration is to be performed, $\overrightarrow{da} = \hat{n} da$, da is infinitesimal area on S and $\hat{n}$ is a unit vector normal to positive side of da. Here $\hat{n}$ must be the outward drawn normal if S is a closed surface. A surface which encloses finite volume will be called as closed surface.



(a) Open surface **(b) Closed surface**

Fig. 1.4: Relation of Normal $\hat{n}$ to a Surface

If S is not closed and is finite then it has a boundary, i.e., it is bounded by a regular curve C and then $\hat{n}$ is to be considered as shown in Fig. 1.4(a). The positive sense of the normal $\hat{n}$ is the direction in which a right hand screw would advance if rotated in the direction of the positive sense on the boundary curve.

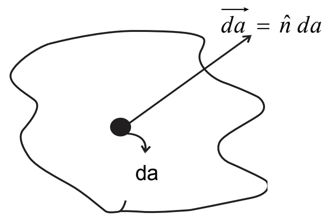


Fig. 1.5: Vector Surface Area

Consider Fig. 1.5, S is closed then the surface integral of V, is written as,

$$\oint_S \vec{V} \cdot \hat{n}\, da$$

In component form, we get

$$\int_S \vec{V} \cdot \hat{n}\, da = \hat{i}\int_S V_x da_x + \hat{j}\int_S V_y da_y + \hat{k}\int_S V_z da_z$$

Where $|da_x| = dydz$, $|da_y| = dzdx$ and $|da_z| = dxdy$

Thus, surface integral is a scalar and depend on surface S. In physics we come across surface integrals of the type, (i) $\int_S \phi\, da$, (ii) $\int_S \vec{V} \cdot \overrightarrow{da}$ and (iii) $\int_S \vec{V} \times \overrightarrow{da}$.

1.2.3 Volume Integrals

Volume integral is an integral over a three dimensional domain.

Let $\phi(x, y, z)$ be a single valued function defined through a three dimensional region volume τ. Divide τ into a large number of sub-regions each of small volume $d\tau_i$. For each element an interior point is chosen and the value of $\phi_i(x, y, z)$ at that point found. The product of each volume element $d\tau_i$ with corresponding value of $\phi_i(x, y, z)$ is found and sum of these computed.

Thus, if $\vec{F}$ is a vector field and ϕ is a scalar point function, the volume integral can be written as,

$J = \oint_\tau \phi \, d\tau$ and $\vec{K} = \oint_\tau \vec{F} \cdot \vec{d\tau}$

Where, $d\tau$ is an infinitesimal volume element.

Note: The volume integral is the usual three dimensional Riemann integral. This integrals can be evaluated in cartesian coordinates. Sometimes it is convenient to evaluate them in curvilinear co-ordinate system.

In cartesian coordinate system,

$d\tau = dx\, dy\, dz$

In cylindrical coordinate system,

$d\tau = (dr)\,(r d\phi)\, dz$

$= r\, dr\, d\phi\, dz$

In spherical polar coordinate system,

$d\tau = (dr)\,(r\, d\theta)\,(r \sin\theta\, d\phi)$

$= r^2 \sin\theta\, dr\, d\theta\, d\phi.$

1.3 Ordinary Integration

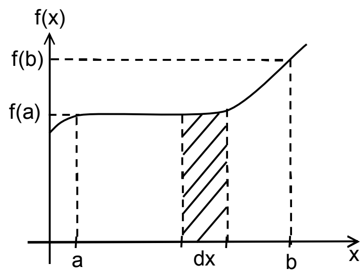


Fig. 1.6

Let $f(x)$ is a function of one variable then fundamental theorem of **calculus** states that for a function $f(x)$, the integral of a derivative of $f(x)$ over an interval.

$$\int_a^b F(x)dx = f(b) - f(a) \text{, where, } F(x) = \frac{df}{dx}$$

OR

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

Geometrical Interpretation

$df = \left(\frac{df}{dx}\right) dx$ tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount dx. The fundamental theorem of calculus says that if you divide the interval from a to b into many tiny divisions dx and add up the increments df from each, the result will be equal to $f(b) - f(a)$.

1.3.1 The Fundamental Theorem for Gradients

Let $F(x, y, z)$ be any scalar function. Starting at point $a = (a_x, a_y, a_z)$, we move a small distance dl_1, as shown in Fig. 1.7. The function will change by an amount,

$$d\vec{F} = (\nabla\vec{F}) \cdot dl_1,$$

Now, we move further by small distance dl_2 then the change in F is $\nabla\vec{F} \cdot dl_2$. In this manner continue till $b = (b_x, b_y, b_z)$.

Hence, total change in F in going from a to b is,

$$\int_{\substack{a \\ \text{line}}}^{b} (\nabla\vec{F}) \cdot \vec{dl} = F(b) - F(a)$$

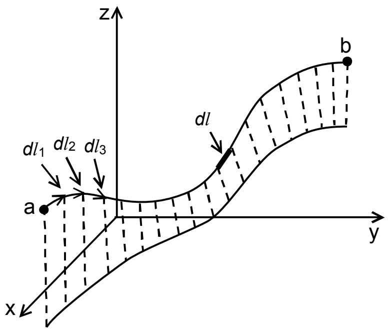


Fig. 1.7

This is the fundamental theorem for gradient. It says that the line integral or path integral along some selected curve of a derivative is given by the value of the function at the boundaries (a, b).

Corollary 1: $\int_a^b (\nabla\vec{F}) \cdot \vec{dl}$ is independent of path followed.

2: $\oint_a^b (\nabla\vec{F}) \cdot \vec{dl} = 0$

If means the line integral over a closed path is zero, i.e., it vanishes.

1.3.2 The Fundamental Theorem for Divergences

Let $\vec{F}$ be a continuous and differentiable vector point function in the region R of space, of volume V bounded by closed surface S. The $\vec{F}$ is a vector point function of position with continuous first derivatives. Then,

$$\int_V (\nabla \cdot \vec{F})\, dV = \oint_S \vec{F} \cdot \vec{da}$$

Here volume integral is extended throughout V and the surface integral is taken over the entire surface S. The term on right, the integral of the normal component of the vector $\vec{F}$ over surface S is sometimes called the **flux** of the vector field $\vec{F}$ through S. The right hand side of the above equation involves the values of $\vec{F}$ on the surface S whereas the left hand side involves the values of $\vec{F}$ throughout the volume V enclosed by S. The surface S is a closed surface.

The theorem is also known as **Gauss's theorem**, **Green's theorem** or simply **divergence theorem**.

Statement: The line integral of divergence of a vector $(\nabla \cdot \vec{F})$ throughout the volume V is equal to the surface integral of the normal component of the vector function over the closed surface bounding the volume V.

Significance of the Theorem

This theorem transforms volume integral into surface integral and *vice-versa*. Divergence theorem is suitable for evaluating surface integrals.

Geometrical Interpretation

Let us consider the flow of in-compressible fluid. At an given point in the volume, the quantity $\nabla \cdot \vec{F}$ or $\text{div} \cdot \vec{F}$ is the outgoing flux per unit volume per unit time, i.e., mass per unit volume that flows through S in unit time in the direction of $\hat{n}$. The $\hat{n}$ is outward unit normal vector of closed surface S which encloses the the volume V.

The flux $d\phi$ of a vector $\vec{F}$ through an infinitesimal surface da is defined as,

$$d\phi = \vec{F} \cdot \vec{da}$$

$\therefore$ For finite surface, the total flux over entire surface is,

$$\phi = \int_S \vec{F} \cdot \vec{da}$$

The total mass of fluid which flows through S from volume V to the outside per unit time in the direction of $\hat{n}$ is given by,

$$\int_S \vec{F} \cdot \vec{da} = \int_S \vec{F} \cdot \hat{n}\, da$$

The boundary of a line is just two end points but the boundary of a volume is an entire continuous area. The $\vec{da}$ represents an infinitesimal element area. It is a vector, whose magnitude is the area of the element and whose direction is normal to the surface, pointing outward.

Example: Electrostatics provides an important application of Gauss's or divergence theorem. For a spherical symmetry, Gauss's law in electrostatics is,

$$\int_S \vec{E} \cdot \vec{da} = \int_S \vec{E} \cdot \hat{n}\, da = \frac{Q}{\varepsilon_0} \int_V S\, dV$$

Using divergence theorem,

$$\int_V \nabla \cdot \vec{E}\, dV = \int_S \vec{E} \cdot \hat{n}\, da$$

$$\therefore \int_V \nabla \cdot \vec{E}\, dV = \frac{Q}{\varepsilon_0} \int_V dV$$

$$\therefore \nabla \cdot \vec{E} \int_S dV = \frac{Q}{\varepsilon_0} \int_V dV$$

$$\therefore \nabla \cdot E = \frac{Q}{\varepsilon_0}$$

This is differential form of Gauss' law in electrostatics.

1.3.3 The Fundamental Theorem for Curl

If *S* is an open, two sided surface bounded by a closed, non-intersecting curve *C* and if a vector function *F* has continuous first partial derivetives then,

$$\int_S (\nabla \times \vec{F}) \cdot \vec{da} = \int_S (\nabla \times \vec{F}) \cdot \hat{n}\, da = \oint_c \vec{F} \cdot \vec{dl}$$

dl is an infinitesimal line element of *C*. $\hat{n}$ unit normal vector outward to surface *S*.

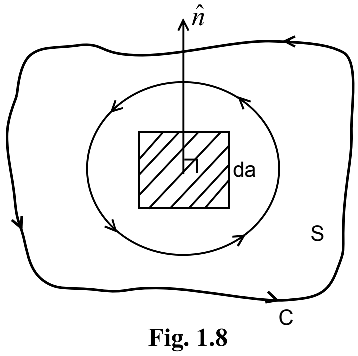


Fig. 1.8

Statement: The line integral of a vector *F* around a closed curve *C* is equal to the integral of the normal component of its curl of *F* over any surface *S* bounded by the curve *C*.

OR

The integral of curl of a vector field '$\vec{F}$' over a surface equals the line integral of vector field over the close curve '*C*' bounding the surface '*S*'.

This theorem is also known as **Curl theorem** or **Stoke's theorem**. Fig. 1.8 shows schematic representation of Curl theorem.

This theorem allows us, (i) to interpret curl as vector source of field and (ii) transforms surface integral into line integral and *vice-versa*.

Geometrical Interpretation: The surface *S* that we have considered is an open surface since there is no inside and outside associated with the surface. We define the positive normal to the surface as follows:

The positive sense of contour *C* is clockwise whenever we look through the surface *S* in the direction of positive normal, i.e., the positive direction of *C* is defined to be that in which an observer on the positive side of *S* would travel to have the interior of *S* on his left and with his head pointing in the direction of positive normal to *S* or other way round *f* or open surface *S* the direction of $\vec{da}$ is given by right hand rule; if your fingers points in the direction of the line integral, then your stretched thumb fixes the direction of $\vec{da}$.

The curl of $\vec{F}$ is integrated over surface area *S* and the line integral of $\vec{F}$ is taken around the periphery of the same area *S*. Thus, Stoke's theorem relates the curl of a vector field $\vec{F}$ inside a contour *C* to the circulation of $\vec{F}$ along the contour *C*.

Physical interpretation: The line integral $\int_C \vec{F} \cdot \vec{dl}$ is called net circulation integral for the vector field $\vec{F}$ around the contour *C* and measures the tendency of the flow lines to circulate.

Consider the example of fluid flow field $\vec{F}$ is fluid velocity. The line integral gives the actual circulation of the fluid around the contour *C*. The region of net circulation is called vortex region.

Example: Magnetostatics provides important application of curl theorem.

According to ampere law,

$$\int_C \vec{B} \cdot \vec{dl} = \mu_0 I_{enc} = \mu_0 I = \mu_0 \int_S J\, da \quad \left(\because I = \int_S J\, da \right)$$

Applying Curl theorem,

$$\int_C \vec{B}\cdot\vec{dl} = \int_S (\nabla\times\vec{B})\cdot\vec{da}$$

$$\therefore \int_S (\nabla\times\vec{B})\cdot\vec{da} = \int_S \mu_0 J\,\vec{da}$$

$$\therefore (\nabla\times\vec{B})\cdot\int_S \vec{da} = \int_S \mu_0 J\,\vec{da}$$

$\therefore (\nabla\times\vec{B}) = \mu_o J$, this is a differential form of Ampere law.

SOLVED PROBLEMS

1.1. Given the vector function $\vec{V} = xy\hat{i} + (x^2+y^2)\hat{j}$. Find the value of the line integral $\int_C \vec{V}\cdot\vec{dl}$,

where, C is arc of $y = x^2 - 4$ from (2, 0) to (4, 12) in xy plane

Solution: $\int_V \vec{V}\cdot\vec{dl} = \int_C [xy\hat{i} + (x^2+y^2)\hat{j}]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$ $\quad\because \vec{dl} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$$\therefore \int_V \vec{V}\cdot\vec{dl} = \int_C [xy\,dx + (x^2+y^2)dy]$$

Along the curve $y = x^2 - 4$ or $x^2 = y + 4$, we get

$$\int_C \vec{V}\cdot\vec{dl} = \int_2^4 (x^3 - 4x)dx + \int_0^{12} (y^2 + y + 4)dy$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_2^4 + \left[\frac{y^3}{3} + \frac{y^2}{2} + 4y\right]_0^{12}$$

$$= 732$$

1.2. Verify Stoke's theorem for the function $\vec{F} = x(x\hat{i} + y\hat{j})$ integration round the square in the plane $z = 0$, whose sides are along the lines $x = 0$, $y = 0$, $x = a$, $y = a$

Solution: $\int_C \vec{F}\cdot\vec{dl} = \int_{OA} \vec{F}\cdot\vec{dl} + \int_{AB} \vec{F}\cdot\vec{dl} + \int_{BC} \vec{F}\cdot\vec{dl} + \int_{CO} \vec{F}\cdot\vec{dl}$

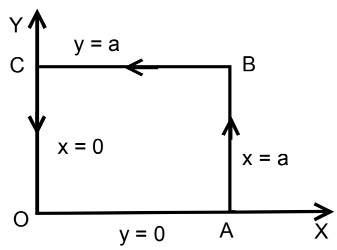


Fig. 1.9

For path OA, $y = 0$, $dy = 0$, $dz = 0$ x changes from 0 to a

$$\therefore \int_{OA} \vec{F}\cdot\vec{dl} = \int_0^a x(x\hat{i} + y\hat{j})(dx\hat{i}) = \int_0^a x^2 dx = \frac{a^3}{3}$$

For path AB, $x = a$, $dx = 0$ $dz = 0$, y changes from $0 \to a$

$$\therefore \int_{AB} \vec{F}\cdot\vec{dl} = \int_0^a x(x\hat{i} + y\hat{j})\cdot(dy\,\hat{j}) = \int_0^a xy\,dy = \int_0^a ay\,dy = \frac{a^3}{2}$$

For path BC, $y = a$, $dy = 0$, $dz = 0$, x changes from a to 0

$$\therefore \int_{BC} \vec{F}\cdot\vec{dl} = \int_a^0 x(x\hat{i} + y\hat{j})\cdot(dx\,\hat{i}) = -\int_0^a x^2 dx = \frac{-a^3}{3}$$

For path CO, $x = 0$, $dx = 0$ $dz = 0$, y changes from a to 0

$$\int_{CO} \vec{F} \cdot \vec{dl} = \int_a^0 x(x\hat{i} + y\hat{j}) \cdot (dy\,\hat{j}) = 0$$

$$\therefore \oint_C \vec{F} \cdot \vec{dl} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

Now, Curl $\vec{F} = \nabla \times [x(x\hat{i} + y\hat{j})]$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$F_x = x^2, \frac{\partial F_x}{\partial y} = 0, F_y = xy \quad \therefore \frac{\partial F_y}{\partial x} = y \text{ and } F_z = 0$$

$$\therefore \text{ Curl } \vec{F} = \frac{\partial F_y}{\partial x} k = yk$$

$$ds = k \cdot dx \cdot dy$$

$$\therefore \int_S (\text{Curl } \vec{F}) \cdot ds = \int_0^a \int_0^a (yk) \cdot (k\,dx\,dy)$$

$$= \int_0^a \int_0^a y\,dx\,dy = \int_0^a \left[\int_0^a y\,dy \right] dx$$

$$= \int_0^a \frac{a^2}{2} dx = \frac{a^2}{2} \int_0^a dx = \frac{a^2}{2} \cdot (a) = \frac{a^3}{2}$$

$\therefore$ The Stoke's theorem is verified.

1.3. Calculate the line integral of the function $F = (xy + y^2)\,\hat{i} + (x^2 + y^2)\,\hat{j}$ along two different paths in xy plane connecting the points $O(0, 0)$ and $P(2, 1)$ the path I: (i) along x-axis (0, 0) to (2, 0) and then parallel to y-axis (2, 0) to (2, 1), path II: along direct straight line $OP(0, 0)$ to (2, 1).

Solution: Path I: (i) along the path OQ : $y = 0$, $\therefore$ $dy = 0$, $dz = 0$

x changes from 0 to 2

$$\therefore \int_{OQ} \vec{F} \cdot \vec{dl} = \int_{OQ} [(xy + y^2)\hat{i} + (x^2 + y^2)\hat{j}](dx\,\hat{i})$$

$$= \int_0^2 (xy + y^2) dx = 0$$

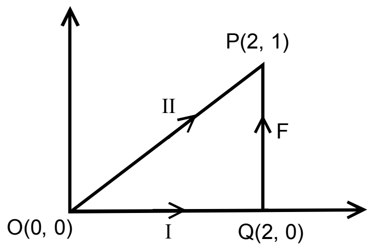


Fig. 1.10

(ii) Along the path QP: $x = 2$, $dx = 0$, $dz = 0$, y changes from 0 to 1

$$\therefore \int_{QP} \vec{F} \cdot \vec{dl} = \int_{QP} [(xy + y^2)\hat{i} + (x^2 + y^2)\hat{j}] \cdot (dy\,\hat{j})$$

$$= \int_0^1 (x^2 + y^2)dy = \int_0^1 (2^2 + y^2)dy = \int_0^1 (4 + y^2)dy$$

$$= \left[4y + \frac{y^3}{3}\right]_0^1 = 4 + \frac{1}{3} = \frac{13}{3}$$

$$\therefore \int_C \vec{F} \cdot \vec{dl} = \int_{OQ} \vec{F} \cdot \vec{dl} + \int_{QP} \vec{F} \cdot \vec{dl}$$

$$= 0 + \frac{13}{3} = \frac{13}{3}$$

Part II: Here equation of line OP is $y = \frac{x}{2}$, $dz = 0$, x changes $x = 0$ to 2, y changes from $y = 0$ to $y = 1$

$$\therefore I = \int_C \vec{F} \cdot \vec{dl} = \int_{OP} [(xy + y^2)\hat{i} + (x^2 + y^2)\hat{j}] \cdot [dx\,\hat{i} + dy\,\hat{j}]$$

$$= \int_{OP} [(xy + y^2)dx + (x^2 + y^2)dy]$$

$$= \int_{OP} (xy + y^2)dx + \int_{OP} (x^2 + y^2)dy$$

$$= \int_0^2 \left[x\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2\right] dx + \int_0^1 ((2y)^2 + y^2)\,dy$$

$$= \int_0^2 \left(\frac{x^2}{2} + \frac{x^2}{4}\right) dx + \int_0^1 (4y^2 + y^2)\,dy$$

$$= \frac{3}{4}\int_0^2 x^2 dx + 5\int_0^1 y^2 dy$$

$$= \frac{3}{4}\left(\frac{x^3}{3}\right)_0^2 + 5\left(\frac{y^3}{3}\right)_0^1$$

$$= \frac{3}{4}\left(\frac{8}{3}\right) + 5\left(\frac{1}{3}\right) = 2 + \frac{5}{3}$$

$$= \frac{11}{3}$$

1.4. Show that surface integral $\oint_S \vec{F} \cdot ds = 6V$, where, S is closed surface enclosing the volume V, $\vec{F} = x\hat{i} + 2y\,\hat{j} + 3z\,\hat{k}$.

Solution: By Gauss's Theorem,

$$\oint_S \vec{F} \cdot \vec{ds} = \int_V (\nabla \cdot \vec{F})\,dV$$

$$\oint_S \vec{F}\cdot\vec{ds} = \int_V \left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(xi + 2y\,\hat{j} + 3z\,\hat{k})\right] dV$$

$$= \int_V \left[\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= \int_V (1+2+3)\,dV = 6\int dV = 6V$$

1.5. Show that $\int_S \vec{F}\cdot\vec{ds} = \frac{3}{2}$,

where, $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$. S is bounded by planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution:

(i) For surface $OABC$, $\vec{ds} = -dx\,dy\,\hat{k}$, $z = 0$, $dz = 0$

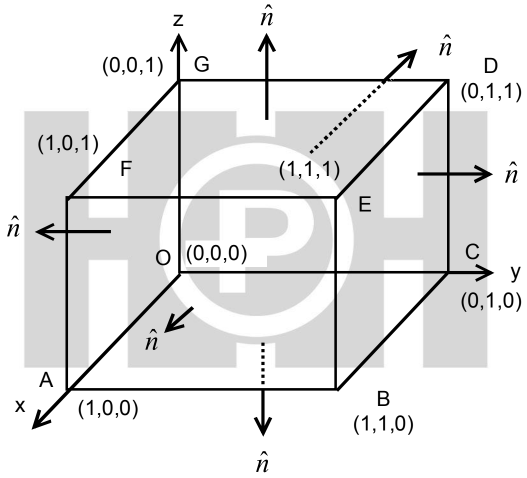


Fig. 1.11

$$\int_{OABC} \vec{F}\cdot\vec{ds} = \int_{OABC} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-dx\,dy\,\hat{k})$$

$$= \int_0^1\int_0^1 (-yz\,dx\,dy) = 0 \qquad (\because z = 0)$$

(ii) For surface $DEFG$, $\vec{ds} = -dx\,dy\,\hat{k}$, $z = 1$, $dz = 0$

$$\int_{DEFG} \vec{F}\cdot\vec{ds} = \int_{DEFG} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(dx\,dy\,\hat{k})$$

$$\int_{DEFG} yz\,dx\,dy = \int_0^1\int_0^1 y(1)\,dx\,dy$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

(iii) For surface $OAFG$, $\overrightarrow{ds} = -dx\,dz\,\hat{j}$, $y = 0$, $dy = 0$

$$\int_{OAFG} \vec{F}\cdot\overrightarrow{ds} = \int_{OAFG}\left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)\cdot\left(-\,dxdz\,\hat{j}\right)$$

$$= \int_{OAFG} y^2 dxdz = 0 \qquad (\because y = 0)$$

(iv) For surface $BCDE$, $\overrightarrow{ds} = dx\,dz\,\hat{j}$, $y = 1$, $dy = 0$

$$\int_{BCDE} \vec{F}\cdot\overrightarrow{ds} = \int_{BCDE}\left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)\cdot\left(dxdz\,\hat{j}\right)$$

$$= \int_{BCDE}(-y^2)dxdz = -\int_0^1 dx\int_0^1 dz = -1$$

(v) For $OCDG$, $\overrightarrow{ds} = -dy\,dz\,\hat{i}$, $x = 0$, $dx = 0$

$$\int_{OCDG} \vec{F}\cdot\overrightarrow{ds} = \int_{OCDG}\left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)\cdot\left(-dy\,dz\,\vec{i}\right)$$

$$= 0$$

(vi) For $ABEF$, $\overrightarrow{ds} = dy\,dz\,\hat{i}$, $x = 1$, $dx = 0$

$$\int_{ABEF} \vec{F}\cdot\overrightarrow{ds} = \int_{ABEF}\left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)\cdot\left(\hat{i}\,dydz\right)$$

$$= \int_{ABEF} 4(1)z\,dy\,dz = \int_0^1\int_0^1 4(1)z\,dy\,dz = 2$$

Now adding all,

$$\int_s \vec{F}\cdot\overrightarrow{ds} = 0 + \frac{1}{2} + 0 - 1 + 0 + 2 = \frac{3}{2}$$

1.6. If $\oint_C \vec{A}\cdot\overrightarrow{dl} = \oint_S \vec{B}\cdot\overrightarrow{ds}$, show that $\vec{B}$ = curl $\vec{A}$

Solution: Given, $\oint_C \vec{A}\cdot\overrightarrow{dl} = \oint_S \vec{B}\cdot\overrightarrow{ds}$

By Stoke's law, $\oint_C \vec{A}\cdot\overrightarrow{dl} = \oint_S \nabla\times\vec{A}\cdot\overrightarrow{ds}$

$\therefore \oint_S \nabla\times\vec{A}\cdot\overrightarrow{ds} = \oint_S \vec{B}\cdot\overrightarrow{dl}$

$\therefore \vec{B} = \nabla\times\vec{A}$ = Curl A

1.7. Test the divergence theorem for the function $\vec{F} = xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}$.. Take it to consideration the volume of cube with sides of length 2 unit and situated at the origin in the first quadrant.

Solution:

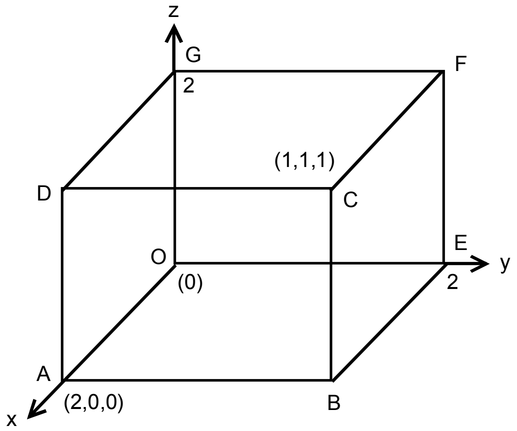


Fig. 1.12

$$\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(xy\hat{i} + 2yz\hat{j} + 3zx\hat{k} \right)$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz)$$

$$= y + 2z + 3x$$

From Divergence theorem,

$$\int_V \nabla \cdot \vec{F}\, dV = \oint_S \vec{F} \cdot ds$$

$$\text{LHS} = \int_V \nabla \cdot \vec{F}\, dV = \int (y + 2z + 3x) dx\, dy\, dz$$

$$= \int_0^2 \int_0^2 \left[\int_0^2 (y + 2z + 3x) dx \right] dy\, dz$$

$$\text{But } \int_{x=0}^{2} (y + 2z + 3x)\, dx = \left[yx + 2zx + \frac{3x^2}{2} \right]_{x=0}^{2}$$

$$= 2y + 4z + 6$$

$$\text{LHS} = \int_0^2 \left[\int_0^2 (2y + 4z + 6)\, dy \right] dz$$

Now

$$\int_{y=0}^{2} (2y + 4z + 6)\, dy = \left[y^2 + 4zy + 6y \right]_{y=0}^{2}$$

$$= 4 + 8z + 12 = (8z + 16)$$

$$\therefore \text{ LHS} = \int_0^2 (8z + 16) dz = \left(4z^2 + 16z \right)_{z=0}^{2}$$

$$= 16 + 32 = 48 \text{ Unit}$$

To evaluate RHS, we must consider separately the the six side of cube.

(i) Surface $ABCD$, $\vec{ds} = \hat{i} dy\, dz$, $x = 0$, $dx = 0$

$$\int_{ABCD} \vec{F} \cdot \vec{ds} = \int_{ABCD} \left(xy\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k} \right) \cdot \left(\hat{i}\, dy\, dz \right)$$

$$= \int_0^2 \int_0^2 2y\, dy\, dz = \int_0^2 \left[y^2 \right]_0^2 dz = \int_0^2 4\, dz = \left[4z \right]_0^2 = 8$$

(ii) Surface $OEFG$, $\vec{ds} = -\hat{i}\,dy\,dz,\ x = 0,\ dx = 0$

$$\int_{OEFG} \vec{F}\cdot\vec{ds} = \int_{OEFG} \left(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}\right)\cdot\left(-\hat{i}\,dy\,dz\right) = \int_{OEFG} (-xy)\,dy\,dz$$

$$= 0$$

(iii) Surface $BCFE$, $\vec{ds} = -\hat{j}dx\,dz,\ y = 2,\ dy = 0$

$$\int_{BCFE} \vec{F}\cdot\vec{ds} = \int_{BCFE} \left(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}\right)\cdot\left(\hat{j}\,dx\,dz\right)$$

$$= \int_0^2\int_0^2 (2yz)\,dx\,dz = \int_0^2\int_0^2 (4z)\,dx\,dz = \int_0^2 \left[2z^2\right]_0^2 dx = \int_0^2 8\,dx = [8x]_0^2 = 16$$

(iv) Surface $AOGD$, $\vec{ds} = -\hat{j}dx\,dz,\ y = 0,\ dy = 0$

$$\int_{AOGD} \vec{F}\cdot\vec{ds} = \int_{AOGD} \left(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}\right)\cdot\left(-\hat{j}\,dx\,dz\right) = \int_{AOGD} (-2yz)\,dx\,dz$$

$$= 0$$

(v) Surface $OABE$, $\vec{ds} = -\hat{k}\,dx\,dy,\ z = 0,\ dz = 0$

$$\int_{OABE} \vec{F}\cdot\vec{ds} = \int_{OABE} \left(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}\right)\cdot\left(-\hat{k}\,dx\,dy\right) = \int_{OABE} (-3zx)\,dx\,dy$$

$$= 0$$

(vi) Surface $CDGF$, $\vec{ds} = \hat{k}\,dx\,dy,\ z = 2,\ dz = 0$

$$\int_{CDGF} \vec{F}\cdot\vec{ds} = \int_{CDGF} \left(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}\right)\cdot\left(\hat{k}\,dx\,dy\right)$$

$$= \int_0^2\int_0^2 (3zx)\,dx\,dz = \int_0^2\int_0^2 (6x)\,dx\,dz = \int_0^2 \left[3x^2\right]_0^2 dz = \int_0^2 12\,dz = [12\,z]_0^2 = 24$$

$$\text{RHS} = \int_S \vec{F}\cdot\vec{ds} = \int_{ABCD} \vec{F}\cdot\vec{ds} + \int_{OEFG} \vec{F}\cdot\vec{ds} + \int_{BCFE} \vec{F}\cdot\vec{ds} + \int_{AOGD} \vec{F}\cdot\vec{ds} + \int_{OABE} \vec{F}\cdot\vec{ds} + \int_{CDGF} \vec{F}\cdot\vec{ds}$$

$$= 8 + 0 + 16 + 0 + 0 + 24$$

$$= 48$$

$\therefore$ LHS = RHS

1.8. Find the total workdone by the force $\vec{F} = (2y\,\hat{i} + xy\hat{j})N$ in moving a particle along the straight line path from $O(0, 0)$ to $P(2, 1)$ m.

Solution: Workdone $= \int_C \vec{F}\cdot\vec{dl} = \int_C \left(2y\,\hat{i} + xy\,\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$

$$= \int_C \left(2y\,dx + xy\,dy\right)$$

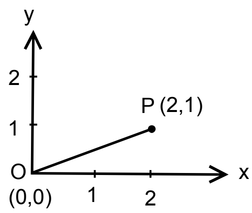


Fig. 1.13

Equation of straight line,

$$y = mx + c$$

$$m = \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{2}$$

$c = 0 = $ y-axis intercept

$$\therefore y = \frac{1}{2}x + 0$$

$$\therefore y = \frac{1}{2}x$$

$\therefore$ The equation of the straight line path is $y = \dfrac{x}{2}$ or $x = 2y$

$$\therefore W = \int_0^2 x\,dx + 2\int_0^1 y^2 dy = \left(\frac{x^2}{2}\right)_0^2 + 2\left(\frac{y^3}{3}\right)_0^1$$

$$= 2 + \frac{2}{3} = \frac{8}{3} = 2.67\,\text{J}$$

1.9. Verfy Stoke's theorem for the function $\vec{F} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$. For the shaded triangular area, shown in Fig. 1.14.

Solution: Let the shaded triangular portion be AOB. The equation of line AB is $y + z = 2$

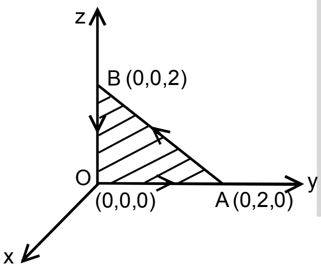


Fig. 1.14

The Stoke's theorem is,

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{ds} = \oint_C \vec{F} \cdot \vec{dl}$$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = -2y\hat{i} - 3z\hat{j} - x\hat{k}$$

Surface element $\vec{ds} = \hat{i}\,dy\,dz,\ x = 0,\ dx = 0$

$$\therefore \int_S (\nabla \times \vec{F}) \cdot \vec{ds} = \int_S (-2y\hat{i} - 3z\hat{j} - x\hat{k}) \cdot (\hat{i}\,dy\,dz)$$

$$= \int_S (-2y)\,dy\,dz - 2\int_0^2 \left[\int_{y=0}^{y=2-z} y\,dy\right] dz$$

$$= -2\int_0^2 \left(\frac{y^2}{2}\right)_0^{2-z} dz = -\int_0^2 (2-z)^2\,dz = -\frac{8}{3}$$

Now infinitesimal displacement is,

$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\therefore \vec{F} \cdot \vec{dl} = (xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= xy\,dx + 2yz\,dy + 3zx\,dx$$

For path OA, $x = 0$, $y : 0 \to 2$, $z = 0$, $dx = dz = 0$

$$\therefore \int_{OA} \vec{F}\cdot\vec{dl} = \int_{OA} 2yz\,dy = 0$$

For path AB, $x = 0, y = 2 \to 0, z = 0 \to 2, dx = 0$

$$\therefore \int_{AB} \vec{F}\cdot\vec{dl} = \int_{AB} 2yz\,dy = \int_2^0 2y(2-y)dy$$

$$= \left[2y^2 - \frac{2y^3}{3}\right]_2^0 = [0] - \left[2\times 4 - \frac{2\times 8}{3}\right] = -\left[8 - \frac{16}{3}\right] = -\frac{8}{3}$$

For path BO, $x = 0, dx = 0, y = 0, dy = 0, z : 2 \to 0$

$$\therefore \int_s \vec{F}\cdot\vec{dl} = \int (xy\,dx + 2yz\,dx + 3zx\,dz) = 0$$

$$\therefore \oint \vec{F}\cdot\vec{dl} = \int_{OA} \vec{F}\cdot\vec{dl} + \int_{AB} \vec{F}\cdot\vec{dl} + \int_{BO} \vec{F}\cdot\vec{dl} = 0 - \frac{8}{3} + 0 = -\frac{8}{3}$$

$\therefore \int_s (\nabla\times F)\cdot ds = \oint \vec{F}\cdot\vec{dl}$, Stroke theorem is verified.

1.10. Check the fundamental theorem for gradient for $\phi = xy^2$ and $a = (0,0,0)$, $b = (2,1,0)$

Solution: The fundamental theorem for gradient is,

$$\int_a^b (\nabla\phi)\cdot\vec{dl} = \phi(b) - \phi(a)$$

Here, $\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ and $\phi = xy^2$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(xy^2\right)$$

$$= y^2\,\hat{i} + 2xy\,\hat{j}$$

Step I: $x : 0 \to 2,\ y = z = 0, dy = dz = 0$

$$\int_I (\nabla\phi)\cdot\vec{dl} = \int_I (y^2\,\hat{i} + 2xy\,\hat{j})\cdot(dx\,\hat{i})$$

$$= \int_0^2 y^2 dx = 0 \qquad (\because y = 0)$$

Step II: $x = 2, y : 0 \to 1,\ z = 0,\ dx = dz = 0$

$$(\nabla\phi)\cdot\vec{dl} = (y^2\,\hat{i} + 2xy\,\hat{j})\cdot(dy\,\hat{j}) = 2xy\,dy = 4y\,dy$$

$$\therefore \int_{II} (\nabla\phi)\cdot\vec{dl} = \int_0^1 4y\,dy = 2$$

$\therefore$ Combining both, we get,

$$\therefore \int_a^b (\nabla\phi)\cdot dl = 0 + 2 = 2$$

Also, $\phi(b) = 2(1)^2 = 2$

$\phi(a) = 0$

$\therefore \phi(b) - \phi(a) = 2$

$\therefore \int_a^b \nabla\phi \cdot \overrightarrow{dl} = \phi(b) - \phi(a)$, hence verified.

1.11. Verify the fundamental theorem for gradients, using $\phi = x^2 + 4xy + 2yx^3$, the points are $A = (0,0,0)$, $B(1,1,1)$ and paths are as shown in Fig. 1.15.

(a) (0,0,0) → (1,0,0) → (1,1,0) → (1,1,1),

(b) (0,0,0) → (0,0,1) → (0,1,1) → (1,1,1),

(c) The parabolic path $z = x^2$, $y = x$.

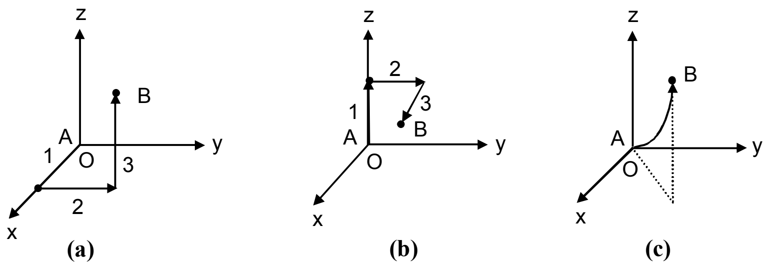


Fig. 1.15

Solution: The fundamental theorem for gradient is,

$$\int_A^B (\nabla\phi) \cdot \overrightarrow{dl} = \phi(B) - \phi(A)$$

Here, $\overrightarrow{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ and $\phi = x^2 + 4xy + 2yz^3$

$$\therefore \nabla\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](x^2 + 4xy + 2yz^3)$$

$$= (2x + 4y)\hat{i} + (4x + 2z^3)\hat{j} + (6yz^2)\hat{k}$$

$\therefore A = (0,0,0)$ and $B = (1,1,1)$

$\therefore \phi(B) = 1^2 + 4(1)(1) + 2(1)(1)^3 = 1 + 4 + 2 = 7$

$\therefore \phi(A) = 0 + 0 + 0 = 0$

$\therefore \phi(B) - \phi(A) = 7 - 0 = 7$ ...(1.1)

Case (a): Refer Fig. 1.15(a),

Segment 1: $x: 0 \to 1,\; y = z = 0,\; dy = dz = 0$

$$\int_{\text{Segment 1}} (\nabla\phi) \cdot \overrightarrow{dl} = \int_0^1 \left[(2x + 4y)\hat{i} + (4x + 2z^3)\hat{j} + (6yz^2)\hat{k}\right] \cdot (dx\;\hat{i})$$

$$= \int_0^1 (2x + 4y)\,dx = \int_0^1 [2x + 4(0)]\,dx$$

$$= \int_0^1 (2x)\,dx = \left[x^2\right]_0^1 = (1^2 - 0) = 1$$

Segment 2: $y : 0 \to 1,\ x = 1$ and $z = 0,\ dx = dz = 0$

$$\int_{\text{Segment 2}} (\nabla\phi)\cdot\overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot(dy\,\hat{j})$$

$$= \int_0^1 (4x + 2z^3)\,dy = \int_0^1 [4(1) + 2(0)]\,dy$$

$$= \int_0^1 (4)\,dy = \left[4y\right]_0^1 = 4$$

Segment 3: $z : 0 \to 1,\ x = y = 1$ and $dx = dy = 0$

$$\int_{\text{Segment 3}} (\nabla\phi)\cdot\overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot(dz\,\hat{k})$$

$$= \int_0^1 (6yz^2)\,dz = \int_0^1 [6(1)z^2]\,dz = \int_0^1 (6z^2)\,dz$$

$$= \left[2z^3\right]_0^1 = 2$$

$\therefore$ Combining all segments, we get,

$$\int_A^B (\nabla\phi)\cdot\overrightarrow{dl} = 1 + 4 + 2 = 7 \qquad ...(1.2)$$

Thus, from equations (1.1) and (1.2), for Case (a), the fundamental theorem for gradients is verified.

Case (b): Refer Fig. 1.15(b),

Segment 1: $z : 0 \to 1,\ x = y = 0$ and $dx = dy = 0$

$$\int_{\text{Segment 1}} (\nabla\phi)\cdot\overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot(dz\,\hat{k})$$

$$= \int_0^1 (6yz^2)\,dz = \int_0^1 [6(0)z^2]\,dz = 0$$

Segment 2: $y : 0 \to 1,\ x = 0$ and $z = 1,\ dx = dz = 0$

$$\int_{\text{Segment 2}} (\nabla\phi)\cdot\overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot(dy\,\hat{j})$$

$$= \int_0^1 (4x + 2z^3)\, dy = \int_0^1 [4(0) + 2(1)^3] = \int_0^1 2dy = [2y]_0^1 = 2$$

Segment 3: $x: 0 \to 1,\ y = z = 1,\ dy = dz = 0$

$$\int_{\text{Segment 3}} (\nabla\phi)\cdot \overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot (dx\,\hat{i})$$

$$= \int_0^1 (2x+4y)\, dx = \int_0^1 [2x + 4(1)]\, dx = \int_0^1 [2x+4]\, dx$$

$$= \left[x^2 + 4x\right]_0^1 = 5$$

$\therefore$ Combining all segments, we get,

$$\int_A^B (\nabla\phi)\cdot \overrightarrow{dl} = 0 + 2 + 5 = 7 \qquad ...(1.3)$$

Thus, from equations (1.1) and (1.3), for Case (b), the fundamental theorem for gradients is verified.

Case (c): Refer Fig. 1.15(c),

For parabolic curve $x: 0 \to 1,\ y = x,\ z = x^2$

$\therefore\ dy = dx$ and $dz = 2x\,dx$

$$\int_{\substack{\text{Parabolic}\\ \text{curve}}} (\nabla\phi)\cdot \overrightarrow{dl} = \int_0^1 \left[(2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}\right]\cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_0^1 \left[(2x+4y)\,dx + (4x+2z^3)\,dy + (6yz^2)\,dz\right]$$

$$= \int_0^1 \left\{\left[2x+4(x)\right]dx + \left[4x + 2(x^2)^3\right]dx + \left[6(x) + (x^2)^2\right]2x\,dx\right\}$$

$$= \int_0^1 \left\{6x\,dx + (4x+2x^6)\,dx + (6x^5)2x\,dx\right\}$$

$$= \int_0^1 \left[6x + 4x + 2x^6 + 12x^6\right]dx$$

$$= \int_0^1 (10x + 14x^6)\,dx$$

$$= \left[5x^2 + 2x^7\right]_0^1$$

$$= \left[5(1)^2 + 2(1)^7\right] - 0$$

$$= 5 + 2 - 0 = 7 \qquad ...(1.4)$$

Thus, from equations (1.1) and (1.4), for Case (c), the fundamental theorem for gradients is verified.

QUESTIONS

1. What are line integral, surface integral and volume integral?
2. What do you mean by conservative field? Give example.
3. State and explain fundamental theorem of gradient.
4. State and explain fundamental theorem of divergence.
5. State and explain fundamental theorem of curl.
6. What do you mean by flux of a vector field? Also, show that flux of a vector field $\vec{F}$ through the surface is $\int_S \vec{F}\cdot\vec{ds}$.

UNSOLVED PROBLEMS

1. Given $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$. The curve C is the rectangle in xy plane bounded by $y = 0, x = a$, $y = b, x = 0$. Find $\int_C \vec{F}\cdot\vec{dl}$ directly. (**Hint:** $\int_C \vec{F}\cdot\vec{dl} = \int_0^a x^2\,dx - 2a\int_0^b y\,dy + \int_a^0 (x^2+y^2)\,dx + \int_b^0 0\,dy$) [**Ans.:** $-2ab^2$]
2. Evaluate the line integral of the function $\vec{F} = (x^2 + 2y)\hat{i} + (x + y^2)\,\hat{j}$ from $A(0,1)$ to $B(2,3)$ along the curve $y = x + 1$. [**Ans.:** $\frac{64}{3}$]
3. Use Gauss's theorem to show that $\frac{1}{3}\int_S \vec{r}\cdot ds = V$ where V is volume enclosed by S, and $\vec{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$ (**Hint:** $\nabla\cdot\vec{r} = 3$)
4. Using Stoke's theorem show that $\int_C \vec{r}\cdot\vec{dr} = 0$ where symbols have usual meaning.
5. Find the workdone by force $\vec{F} = y\hat{i} + x\,\hat{j}$ which displays a particle from (0,0,0) to $(\hat{i} + \hat{j})$ use line integral.
6. If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displays the body in xy plane from (0,0,0) to $P(1,4)$ alongs the curve $y = 4x^2$. Find workdone. [**Ans.:** $W = 20.8$ J]
7. If $\vec{F} = \nabla\times\vec{V}$ then show that $\int_S \vec{F}\cdot\vec{ds} = 0$, where S is any closed surface. (**Hint.:** $\nabla\cdot\vec{F} = \nabla\cdot\nabla\times\vec{V} = 0$)
8. Very divergence theorem using the vector function $\vec{F} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\hat{k}$ and unit cube situated at the origin in 1st octant. [**Ans.:** 3]
9. Suppose $V = (2xz + 3y^2)\hat{j} + (4yz^2)\hat{k}$. Check Stoke's theorem for the surface shown in the Fig. 1.16. [**Ans.:** $\frac{4}{3}$]

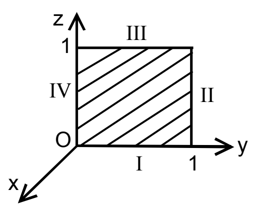


Fig. 1.16

10. Evaluate a surface integral $\int_S \frac{\vec{r}}{r^3}\,\overrightarrow{ds}$, where S denote the surface of sphere of radius 'a' with centre at origin. (**Hint.:** $\hat{r} = \frac{\vec{r}}{r}$, $\overrightarrow{ds} = \hat{r}\,ds$)

    [**Ans.:** 4π]

11. Show that $\int_V (\nabla \times \vec{B})dV = \int_S (\hat{n} \times \vec{B})\overrightarrow{ds}$.

12. Calculate $F = \int (x^2 dy - y\,dx)$ over the,
    - (a) Straight line $y = x$ from (0,0) to (1,1),
    - (b) Parabola $y = x^2$ from (0,0) to (1,1),
    - (c) Integrate round the square, $A(0,0)$, $B(1,0)$, $C(0,1)$, $D(1,1)$. [**Ans.:** $-\frac{1}{6}, -\frac{1}{6}, 2$]

13. Verify divergence theorem for given function, $\vec{F} = (x^3 - yz)\hat{i} + (y^3 - zx)\hat{j} + (z^3 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$ and $0 \le z \le c$.

14. Evaluate $\int_S \vec{F} \cdot \overrightarrow{ds}$, use divergence theorem; if $\vec{F} = 4x\hat{i} - 2y^2\hat{j} - z^2\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. [**Ans.:** 84π]

MULTIPLE CHOICE QUESTIONS

1. Gauss's theorem is used to transform ________.
    - (a) a volume integral into surface integral
    - (b) a volume integral into line integral
    - (c) a surface integral into line integral
    - (d) none of these

2. Stoke's theorem is used to transform ________.
    - (a) a surface integral into volume integral
    - (b) a surface integral into line integral
    - (c) a volume integral into line integral
    - (d) none of these

3. The line integral of a vector field around a closed path can always be written as ________.
    - (a) $\int_S \left(\vec{\nabla} \cdot \vec{F}\right)\overrightarrow{ds}$
    - (b) $\oint_S \left(\nabla \cdot \vec{F}\right) \cdot \overrightarrow{ds}$
    - (c) $\int_S \left(\vec{\nabla} \times \vec{F}\right) ds$
    - (d) $\oint_S \left(\nabla \times \vec{F}\right) \cdot ds$

4. The surface integral of a vector field over the closed surface can be written as ________.
    - (a) $\int_V \left(\vec{\nabla} \times \vec{F}\right) dv$
    - (b) $\oint_V \left(\vec{\nabla} \times \vec{F}\right) \cdot dv$
    - (c) $\int_V \left(\vec{\nabla} \cdot \vec{F}\right) dv$
    - (d) $\oint_V \left(\vec{\nabla} \times \vec{F}\right) dv$

5. The divergence theorem applies to ________.
   (a) time-varying as well as static fields (b) time-varying fields only
   (c) static fields only (d) none of above
6. The mathematical perception of gradient is said to be ________.
   (a) tangent (b) chord
   (c) slope (d) arc
7. The gradient of $x\hat{i} + y\hat{j} + z\hat{k}$ is ________.
   (a) 0 (b) 1
   (c) 2 (d) 3
8. Find the gradient of function given by $x^2 + y^2 + z^2$ at (1, 1, 1).
   (a) $\hat{i} + \hat{j} + \hat{k}$ (b) $2\hat{i} + 2\hat{j} + 2\hat{k}$
   (c) $2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ (d) $4x\hat{i} + 2y\hat{j} + 4z\hat{k}$
9. The gradient of function $\sin x + \sin y$ is ________.
   (a) $\cos x\hat{i} - \sin y\hat{j}$ (b) $\cos x\hat{i} + \sin y\hat{j}$
   (c) $\sin x\hat{i} - \cos y\hat{j}$ (d) $\sin x\hat{i} + \cos y\hat{j}$
10. Compute divergence of the vector $x\hat{i} + y\hat{j} + z\hat{k}$.
    (a) 0 (b) 1
    (c) 2 (d) 3
11. Determine the divergence of $\vec{F} = 30\hat{i} + 2xy\hat{j} + 5xz^2\hat{k}$ at (1, 1, –0.2) and state the nature of field.
    (a) 1, solenoidal (b) 0, solenoidal
    (c) 1, divergent (d) 0, divergent
12. Find whether the rector is solenoidal $\vec{E} = yz\hat{i} + xz\hat{j} + xy\hat{k}$.
    (a) yes, solenoidal (b) no, non-solenoidal
    (c) solenoidal with negative divergence (d) variable divergence
13. Identify the nature of field it the divergence is zero and curl is also zero.
    (a) solenoidal, irrotational (b) divergent, rotational
    (c) solenoidal, rotational (d) divergent, irrotational
14. Which of the following theorem use the curl operation?
    (a) Green's theorem (b) Gauss Div. Theorem
    (c) Stoke's theorem (d) Maxwell's equation
15. The integral form of potential and fixed relation is given by line integral.
    (a) True (b) False
    (c) May be true or false (d) none

16. The potential between two points $P(1, -1, 0)$ and $Q(2, 1, 3)$ with $\vec{E} = 40xy\hat{i} + 20x^2\hat{j} + 2\hat{k}$ is _____.
    (a) 104 (b) 105
    (c) 106 (d) 107
17. The Stoke's theorem uses which of the following operation, ________.
    (a) Divergence (b) Gradient
    (c) Curl (d) Laplacian
18. Evaluate surface integral $\int\int (3x\hat{i} + 2yj)\, ds$ where s is sphere given by $x^2 + y^2 + z^2 = 9$.
    (a) 120 π (b) 180 π
    (c) 240 π (d) 300 π
19. The Gauss divergence theorem converts ________.
    (a) line to surface integral (b) line to volume integral
    (c) surface to line integral (d) surface to volume integral

[Ans.: 1. (a); 2. (b); 3. (c); 4. (c); 5 (a); 6. (c); 7. (d); 8. (b); 9. (a); 10. (d); 11. (b); 12. (a); 13. (a); 14. (c); 15. (a); 16. (c); 17. (c); 18. (b); 19. (d)]

■■■