As per New Syllabus (CBCS Scheme) for Second Semester, B.Sc. (Physics) of Calcutta and other Indian Universities w.e.f. 2018-2019

WAVES AND OPTICS

[With Lab Manual]

Dr. D.C. Tayal

Himalaya Publishing House
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PREFACE

This text, *Waves and Optics* is primarily written as per syllabus (under-CBCS Scheme) for B. Sc. Physics II- semester, Calcutta University. The subject matter has been woven in such a systematic manner that the students may easily be familiar with the concepts of wave and their applications, so as to explain the various phenomena of sound and light.

For the students of Science, it is essential to develop the experimental skill along with the theory, therefore the experiments based on the subject matter of theory and prescribed in the syllabus have been discussed in the third section of the book as - Laboratory Manual.

To check the knowledge & understanding of the subject, a large number of exercises, based on the subject matter, have been placed at the end of each chapter. The multiple choice and short answer types of questions are given at the end of each chapter. The problems & long answer questions are also given for practice purposes.

The first section of the book covers Waves- have been discussed in four chapters. The *first chapter* deals with the Oscillations- Simple harmonic, damped and forced oscillations along with the concept of resonance. The *chapter second* contains the Superposition of harmonic oscillations- Principle of superposition, two and N- collinear harmonic oscillations with equal frequencies, different frequencies. A due weight-age is given to Lissajous figures due to superposition of two perpendicular harmonic oscillations. The *third chapter* consists of Wave Motion –Different types of waves and velocity of transverse vibrations on stretched strings & longitudinal vibrations along the rod and in the fluid in a pipe. The Superposition of harmonic waves are discussed in the fourth chapter, with a detailed study of the standing waves and their applications and the concept of phase and group velocities.

The second section of the text deals with Optics - divided in seven chapters. The first chapter deals with the Wave optics, the second with the Interference, where the coherent sources are obtained by division of wave- front. The third chapter covers the theory of thin plane and wedge shaped films, Newton's rings and their applications are discussed in details. Michelson's and Fabry Perot interferometers are described in the fourth chapter. Fraunhofer and Fresnel's diffractions are discussed in details in the fifth and sixth chapters. Resolving power and applications of Fresnel's integrals have been given special attention. Holography, the modern technique is described in brief in chapter seventh.

The laboratory manual, the third section of the book consists of all the experiments prescribed in the syllabus. They are given in the systematic manner as to follow while experimentation and recording in the practical note book. More stress is given to tell how to get prepared for the lab and to analysis the data and discuss the results with criticisms. .

I would like to thank M/S Himalaya Publishing House for their keen interest in bringing out this separate text for Calcutta University. M/S Times Printographics deserve my appreciations for the nice type setting. Sincere thanks are also for the supporting staff.

Any suggestions, criticisms for the betterment of the subject matter will be deeply appreciated and incorporated with acknowledgement in the future editions.

Greater Noida **DR. D.C. TAYAL**
26th January, 2019

SYLLABUS

2.4. Semester - 2 : Waves and Optics
Waves and Optics (Theory)

Paper : PHS - A - CC - 2 - 4 - TH Credits : 4

1. **Oscillations**

   (*a*) SHM: Simple Harmonic Oscillations. Differential equation of SHM and its solution. Kinetic energy, potential energy, total energy and their time - average values. Damped oscillation. Forced oscillations: Transient and steady states ; Resonance, sharpness of resonance; power dissipation and Quality Factor.

2. **Superposition of Harmonic Oscillations**

   (*a*) Superposition of Collinear Harmonic oscillations: Linearity and Superposition Principle. Superposition of two collinear oscillations having (1) equal frequencies and (2) different frequencies (Beats). Superposition of N collinear Harmonic Oscillations with (1) equal phase differences and (2) equal frequency differences.

   (*b*) Superposition of two perpendicular Harmonic Oscillations: Graphical and Analytical Methods. Lissajous Figures with equal an unequal frequency and their uses.

3. **Wave motion**

   (*a*) Plane and Spherical Waves. Longitudinal and Transverse Waves. Plane Progressive (Travelling) Waves. Wave Equation. Particle and Wave Velocities. Differential Equation. Pressure of a Longitudinal Wave. Energy Transport. Intensity of Wave.

   (*b*) Water Waves: Ripple and Gravity Waves.

4. **Velocity of Waves**

   (*a*) Velocity of Transverse Vibrations of Stretched Strings.

   (*b*) Velocity of Longitudinal Waves in a Fluid in a Pipe. Newton's Formula for Velocity of Sound. Laplace's Correction.

5. **Superposition of Harmonic Waves**

   (*a*) Standing (Stationary) Waves in a String : Fixed and Free Ends. Analytical Treatment. Changes with respect to Position and Time. Energy of Vibrating String. Transfer of Energy. Normal Modes of Stretched Strings. Plucked and Struck Strings. Melde's Experiment.

   (*b*) Longitudinal Standing Waves and Normal Modes. Open and Closed Pipes.

   (*c*) Superposition of N Harmonic Waves. Phase and Group Velocities.

6. **Wave Optics**

   (*a*) Electromagnetic nature of light. Definition and properties of wave front. Huygens Principle. Temporal and Spatial Coherence.

7. **Interference**

(*a*) Division of amplitude and wavefront. Young's double slit experiment. Llyod's Mirror and Fresnel's Biprism. Phase change on reflection : Stokes' treatment. Interference in Thin Films : parallel and wedge-shaped films. Fringes of equal inclination (Haidinger Fringes) ; Fringes of equal thickness (Fizeau Fringes). Newton's Rings: Measurement of wavelength and refractive index.

8. **Interferometers**

(*a*) Michelson Interferometer - (1) Idea of form of fringes (No theory required), (2) Determination of Wavelength, (3) Wavelength Difference, (4) Refractive Index, and (5) Visibility of Fringes.

(*b*) Fabry - Perot interferometer.

9. **Diffraction and Holography**

(*a*) Fraunhofer diffraction : Single slit, Circular aperture, Resolving Power of a telescope. Double slit. Multiple slits. Diffraction grating. Resolving power of grating.

(*b*) Fresnel Diffraction: Fresnel's Assumptions. Fresnel's Half - Period Zones for Plane Wave. Explanation of Rectilinear Propagation of Light. Theory of a Zone Plate: Multiple Foci of a Zone Plate. Fresnel's Integral. Fresnel diffraction pattern of a straight edge, a slit and a wire.

(*c*) Holography: Principle of Holography. Recording and Reconstruction Method. Theory of Holography as Interference between two Plane Waves. Point source holograms.

WAVES AND OPTICS (PRACTICAL)

PHS - A-CC-2-4-P **Credits: 2**

List of Practicals

1. To determine the frequency of an electric tuning fork by Melde's experiment and verify λ^2-T law.
2. To determinate refractive index of the material of a prism using sodium source.
3. To determine the dispersive power and Cauchy constants of the material of a prism using mercury source.
4. To determine wavelength of sodium light using Fresnel Biprism.
5. To determine wavelength of sodium light using Newton's Rings ?
6. To determine the thickness of a thin paper by measuring the width of the interference fringes produced by a wedge-shaped Film.
7. Measurement of the spacing between the adjacent slits in a grating by measuring λ v/s $\sin\theta$ graph of certain order of grating spectra.

CONTENTS

Chapter .... *Page No.*

SECTION A : WAVES

SECTION B : OPTICS

Section A : Waves

1 Oscillations

1.1. INTRODUCTION

Any sort of motion which repeats itself in equal intervals of time is called *periodic motion.* If a particle moves back and forth periodically over the same path, about an equilibrium position, the motion is called *oscillatory* or *vibratory*. Some examples are : simple harmonic motion or the oscillations of a mass attached to a string or of the mass attached to a spring, atoms in molecules or in a lattice, a violin string and a motion of a pendulum. Radio waves and visible light are the oscillating electric and magnetic vectors. The understanding of periodic motion is very much essential for the study of waves, sound, alternating current and light. A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or a torque comes into play to pull it back to the equilibrium position. As soon as it reaches the equilibrium position it gains kinetic energy which is responsible for the forward motion on the other side. When it stops, it is pulled back and thus continues to move to and fro about the position of equilibrium. Oscillation always occurs if the force tends to return the body/system to equilibrium.

As we will see that the displacement of a particle in periodic motion can always be expressed in terms of sines or cosines or a combination of both. It is due to this reason the periodic motion is often called *harmonic motion.* The frequency of harmonic motion ν is the number of cycles/second and is therefore the reciprocal of the period. The unit of frequency in S.I. system of units is *hertz*.

The oscillations often tend to die out with time due to resistive forces. Such oscillations may continue if some periodic force is applied. Such oscillations may be mechanical or electrical depending on the system. An applied periodic force may produce a very large response or greater displacements from equilibrium.

1.2. SIMPLE HARMONIC OSCILLATOR

The periodic motion is bounded between two fixed limits. The particle/body undergoes harmonic motion passes back and forth through its position of stable equilibrium, at which the potential energy is minimum. If a body, which may be assumed as a particle, oscillates between the limits A and B with the equilibrium position O, its potential energy is minimum at O and maximum at the positions A and B. The variation of potential energy is shown in Fig. 1.1(a). The negative gradient of the potential energy at any position gives the force acting on the particle at that position ($F = -dU/dx$). The force is zero at the equilibrium position O. It points to the right when the particle is to the left of the equilibrium position O and has a positive value. It points to the left when the particle is to the right of position O and has a negative value. The variation of force F with the position, $F(x)$, with x is shown in Fig. 1.1(b). Since the force always acts to accelerate the particle towards its equilibrium, hence is called *restoring force.*

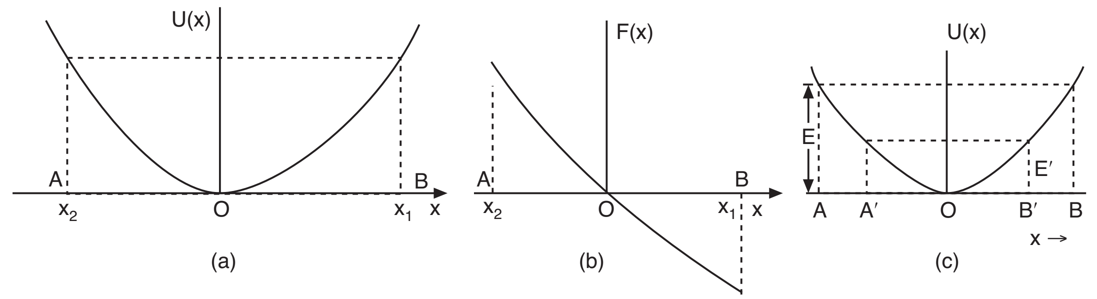

Fig. 1.1. Variation of potential energy and force with displacement.

If there is no dissipation of energy due to frictional or viscous forces, the energy of the oscillating particle remains constant, *i.e.,*

Total energy E = Kinetic energy K + Potential energy U.

This relation shows that the particle cannot move outside the limits A and B, because this would require the negative kinetic energy, which is not possible. At positions A and B, the total energy is potential and at position O, it is kinetic. Total energy of the oscillating body/particle depends upon the initial setting. The variation of total energy varies the limits of oscillation. Fig. 1.1(c) shows that for energy E', the limits become A' and B'.

A particle or a system of particles oscillating back and forth about its equilibrium position due to a restoring force is called *harmonic oscillator*. If the limits A and B are equally spaced about the equilibrium position (*i.e.,* $OA = OB$), the oscillating particle or the system is called *simple harmonic oscillator*. In such a case the motion is called *simple harmonic motion*.

The potential energy of a simple harmonic oscillator varies as $U(x) = \frac{1}{2}kx^2$, where k is a constant. The force acting on the oscillator is given by the equation

$$\mathbf{F}(x) = -\frac{dU}{dx}\hat{x} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)\hat{x} = -k\mathbf{x}. \quad \ldots(1)$$

We know that for an ideal spring compressed or extended by a distance x, the potential energy is $\frac{1}{2}kx^2$ and the force exerted by the spring is given by $F(x) = -kx$. Here k is called the *force constant*. Thus we may conclude that an *ideal spring acts as a simple harmonic oscillator*. The variation of potential energy and the force acting on a simple harmonic oscillator is shown in Fig. 1.2.

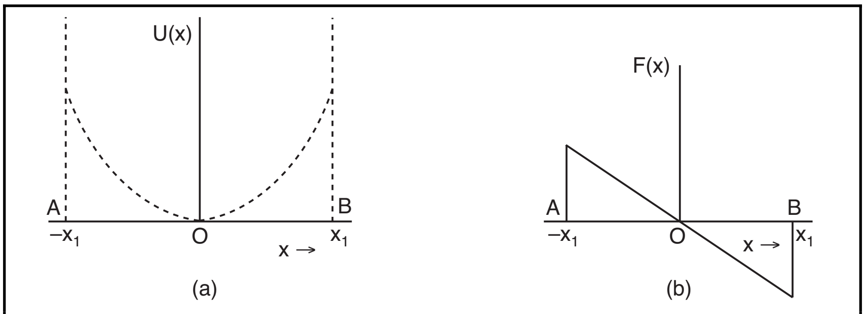

Fig. 1.2. Variation of $U(x)$ and $F(x)$ with x for a simple harmonic oscillator

The relation $\mathbf{F} = -k\mathbf{x}$ is an empirical relation known as Hooke's law for the deformation of elastic bodies. Hooke's law tells us that within the elastic limit, the deformation in the elastic bodies are proportional to the applied forces. Thus we can define *simple harmonic* *oscillator as a system which obeys Hooke's law.* The three dimensional mechanical vibrations may be said to be a combination of simple harmonic oscillators.

1.3. DIFFERENTIAL EQUATION OF SHM AND ITS SOLUTIONS

Simple harmonic motion may be classified as linear or angular according to as the body moves in a straight line about a fixed point under a force or rotates about an axis under a couple/torque. If a body executing *linear simple harmonic motion* is displaced to the left, the force acting on it would point to the right which is given by the relation $\mathbf{F}(x) = -k\mathbf{x}$. Here word linear itself indicates that F is proportional to x rather than to some other power of x.

If $\ddot{x}$ is its acceleration in the direction of x increasing, then from Newton's second law of motion, we have

$$m\,\ddot{\mathbf{x}} = -k\mathbf{x} \quad \text{or} \quad \ddot{x} + (k/m)\,x = 0. \qquad \ldots(2)$$

Here m is the mass of the body and k the force constant.

Similarly, if θ be the angular displacement at any instant from the equilibrium position of the body executing *angular simple harmonic motion* and $\ddot{\theta}$ the angular acceleration in the direction of θ increasing, we have

$$I\,\ddot{\theta} = -c\,\theta \quad \text{or} \quad \ddot{\theta} + (c/I)\,\theta = 0. \qquad \ldots(3)$$

Here I is the moment of inertia of the body about the axis of rotation and c the restoring torque per unit angular displacement.

The equation of motion of a particle executing linear simple harmonic motion, equation (2), may be written as

$$\ddot{x} + \omega^2 x = 0, \qquad \ldots(4)$$

where $\omega^2 = k/m$.

To solve this equation, let $x = A'e^{\alpha t}$, where A' and α are arbitrary constants. Thus $\dot{x} = A'\alpha e^{\alpha t}$ and $\ddot{x} = A'\alpha^2 e^{\alpha t}$. On substituting the values of x and $\ddot{x}$ in equation (4), we get

$$A'\alpha^2 e^{\alpha t} + \omega^2 A' e^{\alpha t} = 0$$

or

$$(\alpha^2 + \omega^2)\,A'e^{\alpha t} = 0.$$

Since $A'e^{\alpha t}$ cannot be zero, otherwise solution will be zero, therefore we have

$$\alpha^2 + \omega^2 = 0 \quad \text{or} \quad \alpha = \pm\, i\omega,$$

where $i = \sqrt{-1}$.

Thus there are two possible solutions $x = A_1'\,e^{i\omega t}$ and $x = A_2'\,e^{-i\omega t}$.

Combining these two solutions of equation (4), the general solution now becomes

$$x = A_1\,e^{i\omega t} + A_2 e^{-i\omega t}, \qquad \ldots(5)$$

where A_1 and A_2 are arbitrary constants.

Expanding the exponentials with the help of Euler's theorem ($e^{\pm i\theta} = \cos\theta \pm i \sin\theta$), above equation may be written as

$$x = A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)$$
$$= (A_1 + A_2)\cos\omega t + i\,(A_1 - A_2)\sin\omega t.$$

Let $A_1 + A_2 = A\sin\delta$ and $i\,(A_1 - A_2) = A\cos\delta$, then we have

$$x = A\sin\delta\cos\omega t + A\cos\delta\sin\omega t$$
$$= A\sin(\omega t + \delta). \qquad \ldots(6)$$

It is the solution of *the equation of* linear simple harmonic oscillator, generally be written as *simple harmonic motion.*

To *find the physical significance of* ω, let us use the relation

$$\sin\theta = \sin(\theta + 2\pi) = \sin(\theta + 4\pi) = \ldots = \sin(\theta + 2n\pi).$$

$$\therefore \quad x = A\sin(\omega t + \delta) = A\sin(\omega t + 2\pi + \delta) = \ldots = A\sin(\omega t + 2n\,\pi + \delta)$$
$$= A\sin(\omega t + \delta) = A\sin[\omega(t + 2\pi/\omega) + \delta] = \ldots = A\sin[\omega(t + 2n\pi/\omega) + \delta].$$

This shows that the displacement x has the same value at the instants, t, $t + 2\pi/\omega$, $t + 4\pi/\omega$, ... $t + 2n\pi/\omega$. It means that the displacement x has the same value after regular interval of $2\pi/\omega$ or the motion of the *simple harmonic oscillator* is periodic and the time period is given by

$$T = 2\pi/\omega = 2\pi\sqrt{m/k}$$
$$= 2\pi/(\text{Acceleration per unit displacement})^{1/2}. \quad \ldots(7)$$

The frequency of the oscillator, which is the number of complete vibrations per unit time, is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \ldots(8)$$

$$\therefore \quad \omega = 2\pi f = 2\pi/T. \quad \ldots(9)$$

The quantity ω, called the *angular frequency*, has the unit of radians/second. Since the displacement x from the equilibrium position ($x = 0$) has the maximum value of A, hence A is called the *amplitude* (the maximum displacement) of the simple harmonic motion. Here we see that the frequency or the time period of the simple harmonic oscillator is independent of its amplitude. After one period T of the motion, the oscillator returns to its initial value or its position repeats itself after each period (T).

The quantity $(\omega t + \delta)$ is called the *phase of motion* and δ the *phase constant* (initial phase angle). The value of δ depends on the displacement and velocity at $t = 0$. The term phase is applied to a vibrating particle in the same sense in which it is applied to the moon. Just as the phases of the moon, *viz.,* crescent, half moon, full moon, etc., tell us about its position and state of motion with respect to the earth, the phase of oscillating object (particle/body) gives us an idea about the position and state of motion of the particle.

1.4. KINETIC, POTENTIAL AND TOTAL ENERGIES

Let us consider a particle of mass m executing linear simple harmonic motion. Its displacement at any instant t may be given by

$$x = A\sin(\omega t + \delta).$$

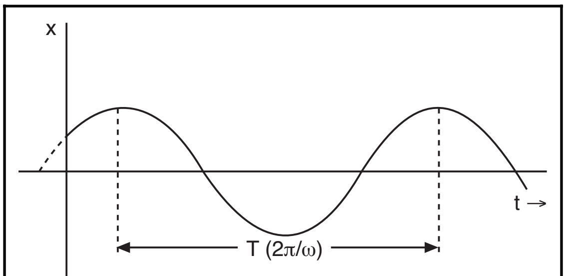

Fig. 1.3. Displacement *v/s* time curve

Its velocity may be obtained as the first time derivative, therefore

$$v = dx/dt = A\omega\cos(\omega t + \delta) = \omega\sqrt{A^2 - x^2}.$$

Its further differentiation will give the acceleration as

$$a = dv/dt = -A\,\omega^2\sin(\omega t + \delta) = -\omega^2 x.$$

Above relations show that at $x = 0$, the mean equilibrium position, velocity of the particle is maximum and the acceleration is zero and at $x = A$, the position of maximum displacement, the velocity is zero and the acceleration is maximum.

The kinetic energy K of the particle at an instant t is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\,A^2\omega^2\cos^2(\omega t + \delta)$$
$$= \frac{1}{2}m\,\omega^2(A^2 - x^2). \quad \ldots(10)$$

The kinetic energy has a maximum value of $\frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2$ at the position of equilibrium. Its value is 0 at $x = A$. It shows that the kinetic energy varies between zero and $\frac{1}{2}kA^2$.

The potential energy U at any instant t is given by

$$U = \text{work done against the restoring force}$$

$$= -\int_0^x F\,dx = -\int_0^x (-kx)\,dx = \frac{1}{2}kx^2.$$

$$= \frac{1}{2}kA^2 \sin^2(\omega t + \delta). \qquad \text{...(11)}$$

Thus we see that the potential energy has maximum value of $\frac{1}{2}kA^2$ at $x = A$. It varies between zero and this maximum value.

Therefore, the total mechanical energy at any instant t may be given by

$$E = K + U = \frac{1}{2}kA^2\cos^2(\omega t + \delta) + \frac{1}{2}kA^2\sin^2(\omega t + \delta) = \frac{1}{2}kA^2. \qquad \text{...(12)}$$

Thus we see that the total energy of the particle executing linear simple harmonic motion is proportional to the square of the amplitude of the motion. Since the average values of $\sin^2(\omega t + \delta)$ and $\cos^2(\omega t + \delta)$ for a period are each equal* to 1/2, therefore

$$\text{Average kinetic energy} = \text{Average potential energy} = \frac{1}{4}mA^2\omega^2 = \frac{1}{4}kA^2$$

and
$$\text{total energy} = (K)_{av} + (U)_{av} = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}kA^2. \qquad \text{...(13)}$$

Thus we see that the energy of a particle executing linear simple harmonic motion is on the average half kinetic and half potential. The energy is all kinetic at the equilibrium position and all potential at the extreme positions (both sides). The variation of energies with the displacement x from the equilibrium position is shown in Fig. 1.4(a). This figure shows that the total energy of a linear harmonic oscillator remains constant, in the absence of any retarding force such as frictional force or viscous force. The variation of energies with time is shown in Fig. 1.4(b).

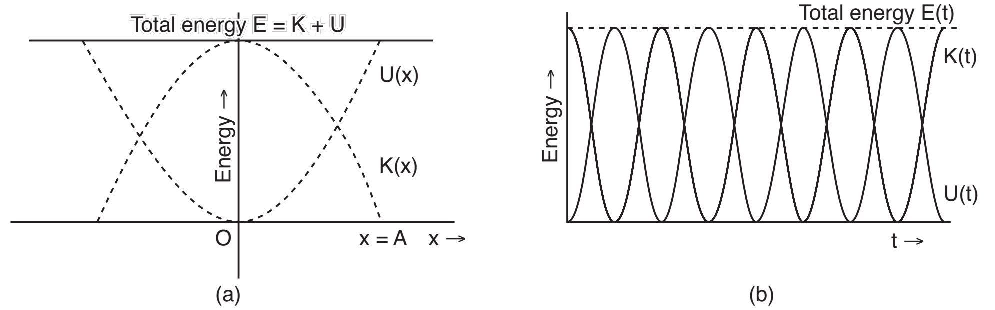

Fig. 1.4. Variation of energy (a) with displacement x and (b) with time t.

The above results also hold in the case of the motion of *angular simple harmonic oscillator*. The similar expressions for its velocity, acceleration and energies can be obtained

---

$$* \ (\sin^2\omega t)_{av} = \frac{1}{T}\int_0^T \sin^2\omega t\,dt = \frac{1}{2T}\int_0^T (1 - \cos 2\omega t)\,dt = \frac{1}{2}$$

$$\text{Similarly } (\cos^2\omega t)_{av} = \frac{1}{T}\int_0^T \cos^2\omega t\,dt = \frac{1}{2T}\int_0^T (1 + \cos 2\omega t)\,dt = \frac{1}{2}.$$

from the corresponding expressions for the linear simple harmonic oscillator by replacing linear displacement x by angular displacement θ and the mass m by moment of inertia I of the oscillator about its axis of rotation.

1.5. DAMPING

In the case of simple harmonic oscillations, it has been assumed that the amplitude of oscillation remains constant, as there is no dissipation of energy. Actually, when the body moves through a retarding medium, such as air, liquid, etc., a part of its energy is dissipated on account of the friction or viscosity of the medium. This energy appears as heat, either in the body itself or in the surrounding medium and a part is radiated into the medium by waves set up in it. On account of this dissipation of energy, the amplitude of vibration gradually diminishes and body ultimately comes to rest. This *reduction in amplitude of an oscillating body is called damping*. The forces responsible for this reduction are called *damping, resistive or dissipative forces*.

When the motion of the body is slow, *i.e., its velocity is not sufficiently large, the damping or resistive force of the medium is proportional to the velocity* v of the body, *i.e.,*

$$F \propto v \quad \text{or} \quad \mathbf{F} = -\gamma\, \boldsymbol{v} = -\gamma\, d\mathbf{x}/dt, \qquad \ldots(14)$$

where γ is a positive constant, called *damping coefficient of medium* and may be defined as the *resistive force or damping force per unit velocity*. Here negative sign shows that the damping force opposes the velocity of the body.

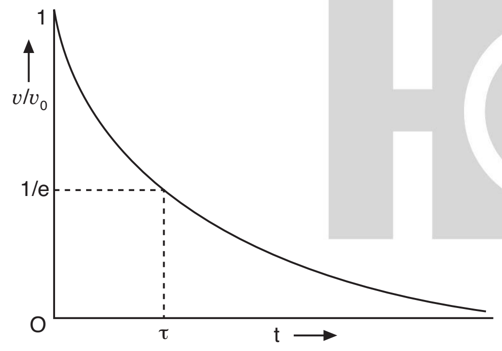

Fig. 1.5. Exponential decrease in velocity due to damping.

If there is no external force acting on the body and its velocity is opposed by the damping force $\mathbf{F}$ only, then from Newton's second law the equation of motion is given by

$$F = m\frac{dv}{dt} = -\gamma\, v \quad \text{or} \quad \frac{dv}{dt} + \left(\frac{\gamma}{m}\right) v = 0, \qquad \ldots(15)$$

where m is the mass of the body. The quantity m/γ is usually denoted by a constant τ and called the *relaxation time*. Thus the above equation may be written as

$$\frac{dv}{dt} = -\frac{1}{\tau}\, v \quad \text{or} \quad \frac{dv}{v} = -\frac{1}{\tau}\, dt.$$

On integration both sides, we get

$$\int \frac{dv}{v} = -\frac{1}{\tau}\int dt \quad \text{or} \quad \log_e v = -(1/\tau)\, t + C,$$

where C is a constant of integration, which may be determined from the initial conditions : at $t = 0$, $v = v_0$. Thus we have $C = \log_e v_0$ and

$$\log_e v - \log_e v_0 = -(1/\tau)\, t$$

$$\text{or} \quad v = v_0\, e^{-t/\tau}. \qquad \ldots(16)$$

This relation shows that the velocity decreases with time exponentially, Fig. 1.5, *i.e., the velocity has been damped with time constant* τ.

At $t = \tau$, $v/v_0 = 1/e = 1/2.718 = 0.368$,

hence the *relaxation time or time constant* τ *may be defined as the time during which the velocity reduces to* $1/e^{th}$ *of its initial value.*

Since the kinetic energy of a particle of mass m moving with velocity v is given by $K = \frac{1}{2} mv^2$, hence the instantaneous value of kinetic energy of the body moving with velocity given by the equation (16) is

$$K.E. = \frac{1}{2} mv_0^2 e^{-2t/\tau} = K_0 e^{-2t/\tau}, \quad ...(17)$$

where $K_0 = \frac{1}{2} mv_0^2$ = the initial kinetic energy of the body. Thus we see that the kinetic energy of the oscillating body falls rapidly with time, with time constant half that for velocity (*i.e.,* time constant = $\tau/2$). Putting $v = dx/dt$ in equation (16), we get

$$v = dx/dt = v_0 e^{-t/\tau} \text{ or } \int dx = \int v_0 e^{-t/\tau} dt$$

$\therefore$ $x = v_0(-\tau) e^{-t/\tau} + C'$,

Since $x = 0$ at $t = 0$, therefore $C' = v_0\tau$,

$$\therefore \quad x = v_0\tau(1 - e^{-t/\tau}). \quad ...(18)$$

This relation shows that at $t \to \infty$, $x \to v_0\tau$, *i.e.,* the maximum distance may be covered in a very large time. It is same as the distance traversed in time τ with the initial velocity v_0.

Let us discuss following two cases of damping.

(1) *Ohmic electrical resistance in LR-circuit.* The damping of similar type is seen in the L-R circuit. The voltage drop across an ideal resistor R is related to the current I by ohm's law as

$$V_R = IR.$$

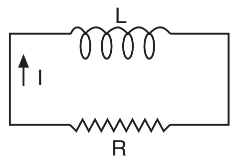

Fig. 1.6. LR-circuit.

On account of the current I, an induced emf is set up across the inductance L. Since there is no external source of emf, hence

$$V_R = V_L.$$

or $IR = -L\, dI/dt$

or $$dI/dt + (R/L) I = 0. \quad ...(19)$$

This relation is identical in form with the relation (15), with $\tau = m/\gamma = L/R$. Therefore its solution will be

$$I = I_0 e^{-(R/L)t}, \quad ...(20)$$

where I_0 is the initial or the maximum value of current. The above equation shows that the current falls exponentially and the resistance is responsible for the damping.

(2) *Mechanical pressure on a moving plate.* Let a flat plate be moving normal to its plane through a gas at low pressure. The speed V of the plate is much slower than the average speed v of the molecules of the gas. If the pressure is so small that we can neglect the collisions of the molecules with each other, then the pressure exerted by the molecules of the gas is proportional to the average momentum transfer per molecule. As we know that the average momentum transfer is proportional to the square of the relative velocity. If P_1 and P_2 be the pressures on the two side, of the plate, then

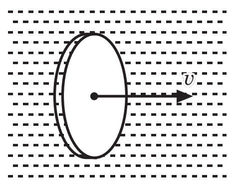

Fig. 1.7. Moving plate in a medium.

$$P_1 \propto (v + V)^2 \text{ and } P_2 \propto (v - V)^2$$

$$\therefore \text{ Net pressure } P = P_1 - P_2 \propto 4vV. \quad ...(21)$$

Thus the net force on the moving plate is proportional to the velocity V of the plate. Its direction is such that it opposes the motion of the plate.

1.6. DAMPED OSCILLATORS

Ideally the harmonic oscillators continue to oscillate with a constant frequency and amplitude. Real world systems always have some dissipative forces and the oscillations thus die out after a certain time, unless we replace the dissipated energy. As discussed earlier, the decrease in amplitude caused by dissipative forces is called *damping* and such a motion is called *damped oscillations*. A harmonic oscillator in which the oscillations are damped on account of dissipative forces is called a *damped harmonic oscillator* and the dissipative force is thus called *damping force*. In the mechanical oscillator, the damping force is due to the effect of friction. Viscous force is a very special type of friction, which arises when an object moves through a fluid either liquid or gas, at speeds which are not so large as to cause turbulence. This force usually is proportional to the velocity of the oscillating body for small oscillations, given by the equation (14).

$$F_x = -\gamma v = -\gamma \, dx/dt.$$

The negative sign shows that this force is always opposite in direction to the velocity.

We know that a restoring force acts on the mass of the body if displaced through a distance x from its equilibrium position, and is given by the relation

$$F_{restoring} = -kx.$$

∴ Net restoring force acting on the body of mass $m = -kx - \gamma \, dx/dt$.

From Newton's second law, the equation of motion is $ma \; (= m\, d^2x/dt^2) = -kx - \gamma\, dx/dt$.

or
$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = 0. \qquad \ldots(22)$$

This is the differential equation for x and is same as that for the linear simple harmonic motion, except for the added term $-\gamma\, dx/dt$. This is still linear. Let us put $\gamma/m = 2b$ and $k/m = \omega_0^2$ in the above equation, which may thus be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0. \qquad \ldots(23)$$

This is called the *differential equation of second order of a damped harmonic oscillator*. Here $2b$ is the damping force per unit mass at an instant when the oscillating body is moving with unit velocity. It is the reciprocal of time τ, called *relaxation time*. b is called *damping constant* or *damping factor*.

Let the solution of equation (23) be of the form

$$x = A\, e^{\alpha t}. \qquad \ldots(24)$$

Substituting the values of x and its derivatives as

$$dx/dt = A\,\alpha\, e^{\alpha t} \text{ and } d^2x/dt^2 = A\,\alpha^2\, e^{\alpha t}$$

in equation (23), we get

$$A\, e^{\alpha t}\,[\alpha^2 + 2b\alpha + \omega_0^2] = 0.$$

As $A\, e^{\alpha t}$ cannot be zero, therefore

$$\alpha^2 + 2b\,\alpha + \omega_0^2 = 0.$$

It is a quadratic equation in α, therefore

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2}.$$

Hence the solutions of equation (23) are :

$$x = A\; e^{(-b+\sqrt{b^2-\omega_0^2})\,t} \text{ and } x = A\; e^{(-b-\sqrt{b^2-\omega_0^2})\,t}$$

These are independent linear solutions, hence the general solution may be written as

$$x = A_1\; e^{(-b+\sqrt{b^2-\omega_0^2})\,t} + A_2\; e^{(-b-\sqrt{b^2-\omega_0^2})\,t}$$
$$= e^{-bt}\,[A_1\, e^{\beta t} + A_2\, e^{-\beta t}], \qquad \ldots(25)$$

where $\beta = (b^2 - \omega_0^2)^{1/2}$. The values of the arbitrary constants A_1 and A_2 depend upon the initial conditions of motion.

Following three cases arise in the solution corresponding to the values of β as real, zero and imaginary.

Case-1, β real or $b > \omega_0$ —When the damping is large compared to the restoring force, *i.e.*, $b > \omega_0$, the indices of e are real and negative.

$$x = A_1 e^{(-b+\beta)t} + A_2 e^{(-b-\beta)t}. \quad \ldots(26)$$

This equation indicates that the displacement, x, after passing its maximum value, decays asymptotically to zero without changing sign. Hence the motion of the body is not oscillatory, but simply one in which the body, after its initial displacement, gradually comes back to its equilibrium position. This type of motion is called *dead beat, aperiodic (non-oscillatory)* or *overdamped*. It has many applications, such as : (*i*) in dead beat galvanometer, (*ii*) in a pendulum oscillating in a viscous fluid, (*iii*) in electrical recording of sound by means of an oscillograph, and (*iv*) in the design of certain forms of sound receivers.

Case-2, $\beta = 0$ or $b = \omega_0$ — If $b = \omega_0$, then the solution becomes

$$x = (A_1 + A_2)\, e^{-bt} = c\, e^{-bt}. \quad \ldots(27)$$

This solution does not represent a general solution of the second order of differential equation, as the latter must have two constants to enable us to specify the initial displacement and initial velocity of the oscillator.

Let us now assume that $(b^2 - \omega_0^2)^{1/2} = h$, which is a very small quantity, but not exactly zero. Thus from equation (25), we have

$$x = e^{-bt}\,[A_1 e^{ht} + A_2 e^{-ht}]$$
$$= e^{-bt}\,[A_1(1 + ht + \ldots) + A_2(1 - ht + \ldots)] = e^{-bt}\,[(A_1 + A_2) + (A_1 - A_2)\, ht + \ldots]$$

Since h is very small, hence neglecting terms containing h^2 and other higher powers of h, we get

$$x = e^{-bt}\,[(A_1 + A_2) + (A_1 - A_2)\, ht] = e^{-bt}\,[P + Qt],$$

where $P = A_1 + A_2$ and $Q = h\,(A_1 - A_2)$.

If initially, the displacement of the body is x_0 and its velocity is v_0, then we have

$$x_0 = P \text{ and } v_0 = Q - bP \text{ or } Q = v_0 + bx_0.$$

$$\therefore \quad x = e^{-bt}\,[x_0 + (v_0 + bx_0)\, t]. \quad \ldots(28)$$

This shows that the displacement x increases with t for small values of t on account of the second term of the bracket, but as the time elapses the exponential term becomes relatively more important and the displacement x returns exponentially from maximum to near zero in the shortest possible time. This type of motion is called *critically damped* or *just aperiodic*. The corresponding value of b is called *critical damping constant*. Thus the necessary condition for critical damping is $b \to \omega_0$. At this stage, the pointer of the moving coil galvanometer moves at once to and stays at the correct position without any oscillations. This type of motion is used in recording sound vibrations either mechanically or electrically.

Case 3, β-imaginary or $b < \omega_0$ — If the damping force is small compared to the restoring force, *i.e.*, $b < \omega_0$, the indices of e are imaginary and we may write

$$\beta = \sqrt{b^2 - \omega_0^2} = i\sqrt{\omega_0^2 - b^2} = i\,\omega,$$

where ω is the real quantity. Thus equation (25) becomes

$$x = A_1 e^{(-b+i\omega)t} + A_2 e^{(-b-i\omega)t}.$$

From Euler's theorem, this may be put in the form of sine and cosine as

$$x = e^{-bt}[A_1(\cos \omega t + i \sin \omega t) + A_2(\cos \omega t - i \sin \omega t)]$$
$$= e^{-bt}\,[(A_1 + A_2)\cos \omega t + i\,(A_1 - A_2)\sin \omega t]$$

Let $A_1 + A_2 = a_0 \sin \theta$ and $i\,(A_1 - A_2) = a_0 \cos \theta$. Thus we have

$$x = a_0\, e^{-bt} \sin(\omega t + \theta). \quad \ldots(29)$$

Here a_0 and θ are the constants to be determined from the initial conditions of the problem. This equation represents a damped oscillatory motion called *under-damped oscillations,* which is simple harmonic in nature whose amplitude $a_0\, e^{-bt}$ decreases exponentially with the time and the time period $T = 2\pi/\omega = 2\pi/\sqrt{\omega_0^2 - b^2}$. The rate of decay depends upon the damping constant b. Since $\omega < \omega_0$, hence the frequency of damped oscillations is smaller than the natural frequency ω_0 due to damping.

Thus we see that the effects of damping forces of the medium on the motion of the body are two fold :

(*i*) *Loss of Amplitude*. The amplitude of the oscillations does not remain constant but decreases exponentially with time, in accordance with the term e^{-bt}. Since the sine term has the maximum values +1 or –1. It alternates in sign at an interval of half of period. Therefore the successive amplitudes a_1, a_2, a_3, ... etc. are separated by half the period.

$$\therefore \qquad \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots\ldots = \frac{a_n}{a_{n+1}} = \frac{a_0\, e^{-bt}}{a_0\, e^{-b\,(t + T/2)}} = e^{bT/2} = d \text{ (say)},$$

where d is called the *decrement of damped harmonic motion*. The quantity $\log_e d$ determines the rate of decay of motion of the body and is called the *logarithmic* decrement. It is usually denoted by λ and is defined as $\lambda = \log_e d$, the natural logarithm of the ratio of two successive amplitudes of oscillations of the body. Thus we have

$$\lambda = \log_e d = bT/2.$$

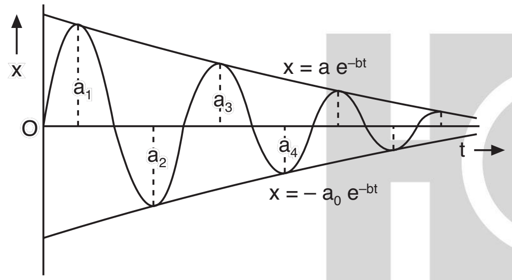

Fig. 1.8. Amplitudes of damped harmonic oscillator.

If a_0 be the true value of amplitude in the absence of damping, then we have

$$\frac{a_0}{a_1} = d^{1/2} = e^{\lambda/2} = \left(1 + \frac{\lambda}{2} + \frac{(\lambda/2)^2}{2\,!} + \ldots\right)$$

$$= 1 + \lambda/2.$$

$$\therefore \quad a_0 = a_1(1 + \lambda/2).$$

and the displacement

$$x = a_1(1 + \lambda/2)\, e^{-bt} \sin(\omega t + \theta) \quad \ldots(30)$$

This shows that x lies entirely between the curves $x = a_0\, e^{-bt}$ and $x = -a_0 e^{-bt}$, as shown in Fig. 1.8.

(*ii*) *Fall of frequency*. The frequency of vibration decreases from its natural frequency $\omega_0/2\pi$ to $(\omega_0^2 - b^2)^{1/2}/2\pi$, hence the time period increases. It is independent of time and hence remains constant throughout the oscillations. However, the change in frequency due to damping effect is negligibly small and may be ignored practically in the case of musical instruments. The effect is considerable if the oscillating body is surrounded by a medium, which exerts a considerable dragging effect.

All the above three types of damped oscillations are shown in Fig. 1.9.

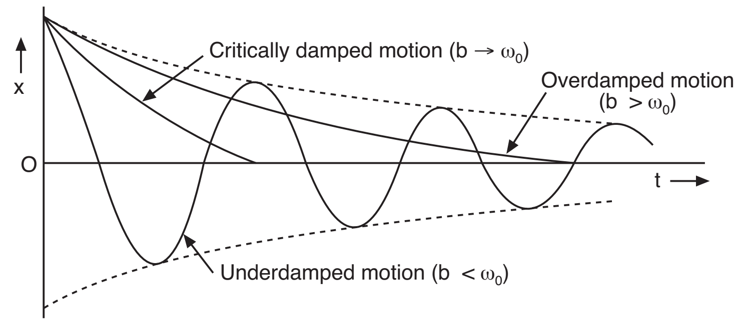

Fig. 1.9. Motion of damped harmonic oscillator.

Power Dissipation. We know that the work is done by the oscillating body overcoming the damping forces of the medium. Therefore the energy of the body continuously dissipates during the oscillations. Let us now calculate the *rate of dissipation of energy,* called the *dissipation power*.

Since the displacement of the body at an instant t is given by

$$x = a_0 e^{-bt} \sin(\omega t + \theta)$$

$$\therefore \quad v = dx/dt = a_0 e^{-bt} [-b \sin(\omega t + \theta) + \omega \cos(\omega t + \theta)].$$

Therefore the kinetic energy at an instant t

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m a_0^2 e^{-2bt} [-b \sin(\omega t + \theta) + \omega \cos(\omega t + \theta)]^2$$

Potential energy of the body at an instant t is given by

$$U = \frac{1}{2} kx^2 = \frac{1}{2} m\omega_0^2 x^2 = \frac{1}{2} m\omega_0^2 a_0^2 e^{-2bt} \sin^2(\omega t + \theta)$$

$\therefore$ Total energy $E(t)$ of the oscillator at an instant t is given by

$$E(t) = \frac{1}{2} m a_0^2 e^{-2bt} [b^2 \sin^2(\omega t + \theta) + \omega^2 \cos^2(\omega t + \theta) - 2b\omega \sin(\omega t + \theta) \cos(\omega t + \theta) + \omega_0^2 \sin^2(\omega t + \theta)]$$

$$\therefore \quad E_{\text{average}} = \frac{1}{T} \int_0^T E(t)\, dt.$$

If the damping is small, the amplitude of the damped harmonic motion may be assumed as constant in one cycle of motion. Hence the factor e^{-2bt} may be taken as constant. Since the average values over a time period T of $\sin^2(\omega t + \theta)$, $\cos^2(\omega t + \theta)$ and $2 \sin(\omega t + \theta) \cos(\omega t + \theta)$ are 1/2, 1/2 and 0 respectively, hence we have

$$E_{\text{average}} = \frac{1}{2} ma_0^2 e^{-2bt} [\tfrac{1}{2} b^2 + \tfrac{1}{2} \omega^2 - 2\, b\omega \times 0 + \tfrac{1}{2} \omega_0^2]$$

$$= \frac{1}{2} ma_0^2 e^{-2bt} \cdot \omega_0^2 = \frac{1}{2} k\, a_0^2 e^{-2bt}. \qquad \text{...(31)}$$

At $t = 0$, the energy of the system

$$E_0 = \frac{1}{2} k\, a_0^2.$$

$$\therefore \quad E_{\text{average}} = E_0 e^{-2bt} = E_0 e^{-t/\tau}. \qquad \text{...(32)}$$

Thus we see that the energy decreases exponentially in time. The decay can be characterised by the time τ required for the energy to drop to $1/e = 0.368$ of its initial value. This time is called the *decay time* or *damping time*. The average power dissipation

$$P_{\text{average}} = dE/dt = 2b\, E_{\text{average}} = E_{\text{average}}/\tau. \qquad \text{...(33)}$$

and loss of energy in one time period $= P_{\text{average}}\, T = E_{\text{average}}\, T/\tau$. ...(34)

Thus we see that this loss of energy is due to the work done against the damping force and usually appears in the form of heat in oscillating system (body + medium).

Quality Factor. The degree of damping of an oscillator is often specified by a dimensionless parameter Q, the quality factor, defined as

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$$

$$= 2\pi \times \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per period}}. \qquad \text{...(35)}$$

This factor measures the quality of a damped harmonic oscillator. The oscillator is of high quality if the damping is small and therefore the higher is its quality factor Q. It is also called the *figure of merit of the oscillator*.

Q is easily calculated for the lightly damped case. The rate of change of energy is from equation (32), as

$$dE/dt = -(1/\tau) E_0 e^{-t/\tau} = -(1/\tau) E.$$

The energy dissipated in a short time interval Δt is the positive quantity, given as

$$\Delta E = |dE/dt|\ \Delta t = (1/\tau)\, E\, \Delta t$$

$\therefore$ Energy dissipated per period = $(1/\tau)\, E.T.$

Therefore the quality factor

$$Q = 2\pi \times \frac{E}{(1/\tau)\, E.T.} = 2\pi \times \frac{\tau}{T} = \omega\tau. \qquad \ldots(36)$$

In case of a low damping $\omega \to \omega_0$ and the quality factor $Q \to \omega_0\tau$.

Since $\omega_0 = \sqrt{k/m}$ and $\tau = m/\gamma$, hence

$$Q = \sqrt{k/m}\ .\ (m/\gamma) = \sqrt{km}/\gamma. \qquad \ldots(37)$$

It shows that for $\gamma \to 0$, $Q \to \infty$, *i.e.,* the lower the damping, the higher the quality.

Equation (32) shows that the average energy of a damped harmonic oscillator reduces to $1/e$ of its initial value in time $t = \tau$. During this time interval oscillator executes $(\omega_0/2\pi)\,\tau = Q/2\pi$ oscillations and hence the phase of the oscillator changes by Q. Thus we may also define *quality factor as the phase change in the oscillator in the time in which the energy of the oscillator reduces to $1/e^{th}$ of its initial value.*

1.7. FORCED HARMONIC OSCILLATOR

We have seen earlier that the oscillations of a harmonic oscillator in a medium get damped with time on account of the damping forces and as a result the amplitude decreases rapidly and the oscillations die out after some time. We can maintain constant amplitude oscillations in the body by applying a sustained periodic force, which has not necessarily the same period as the natural frequency of the body. The energy loss in doing work against damping forces is supplied by the periodic force. In the initial stage the body tends to oscillate with its natural frequency whilst the impressed force tries to impose its own frequency on it. Since the amplitude of the natural oscillations die out in a small time, hence after a small time the body starts oscillating with the constant amplitude and with the frequency of the impressed periodic force. *Such oscillations of constant amplitude and period performed by the body on account of an impressed periodic force are called forced oscillations.* Such a body is called the driven, the impressed force the *driver* and the frequency of the impressed force, the *forcing frequency*. The oscillations of the body are called the *driven* or *forced oscillations* and the body or such a system the *driven* or *forced harmonic oscillator*.

If the frequency of the impressed force is equal to the natural frequency of the oscillations of the body, the amplitude of the oscillations becomes very large. Such oscillations of very large amplitude performed by the body on account of an impressed periodic force of frequency equal to the natural frequency of the body, are called *resonant oscillations* and the phenomenon is called *resonance*. It does not occur if the damping is too large.

Let an oscillating body of mass m be subjected to a periodic force $F = F_0 \sin pt$. Since the damping and restoring forces acting on the body are $-\gamma\, dx/dt$ and $-kx$ respectively. Hence from Newton's law of motion, we have

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx + F_0 \sin pt$$

or

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt, \qquad \ldots(38)$$

where $2b = \gamma/m$, $\omega_0^2 = k/m$ and $f_0 = F_0/m$.

Equation (38) is the differential equation of second order of a forced harmonic oscillator. The general solution of such equations consists of two parts :

(1) *Complementary function*, which is the solution of equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0.$$

The motion is damped out without oscillations for $b \geq \omega_0$. For $b < \omega_0$, the motion is called under damped and is represented by the equation

$$x = a_0 e^{-bt} \sin(\omega t + \theta),$$

where $\omega = \sqrt{\omega_0^2 - b^2}$, thus we see that for $b < \omega_0$, the oscillations die out due to damping after some time on account of the exponential term in the amplitude.

(2) *The particular integral*, which may be assumed to be a periodic function, having frequency as that of the applied periodic force of constant amplitude and frequency, given as

$$x = A \sin(pt - \phi). \qquad \text{...(39)}$$

Substituting the values of x, dx/dt and dx^2/dt^2 from equation (39) in equation (38), we get

$$-Ap^2 \sin(pt-\phi) + 2bAp\cos(pt-\phi) + \omega_0^2 A\sin(pt-\phi) = f_0 \sin pt = f_0\sin(pt - \phi + \phi)$$
$$= f_0[\sin(pt-\phi)\cos\phi + \cos(pt-\phi)\sin\phi].$$

Since this equation must hold for all values of t, hence the coefficients of $\sin(pt - \phi)$ and $\cos(pt - \phi)$ will be same on both the sides of this equation. Thus we have

$$A(\omega_0^2 - p^2) = f_0\cos\phi \qquad \text{...(40)}$$

and $$2b\,Ap = f_0 \sin\phi. \qquad \text{...(41)}$$

Solving these equations, we get

$$A = f_0/\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2} \qquad \text{...(42)}$$

and $$\tan\phi = 2bp/(\omega_0^2 - p^2). \qquad \text{...(43)}$$

Hence the equation (39) becomes

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}}\sin\left(pt - \tan^{-1}\frac{2bp}{\omega_0^2 - p^2}\right). \qquad \text{...(44)}$$

The general solution of equation (38) is thus given by

$$x = a_0e^{-bt}\sin(\omega t + \theta) + \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}}\sin(pt - \phi). \qquad \text{...(45)}$$

The first term, representing the free (damped) oscillations, gives the transient solution. The second term, representing the forced vibrations, gives the steady state solution. The former is effective at the initial stage and the free (damped) oscillations of the body die out when $bt >> 1$ and the body is left to oscillate with a frequency equal to that of the impressed force regardless of the natural (free) period of the body and the amount of damping. However the amplitude and phase depend on the damping and natural frequency.

1.8. TRANSIENT AND STEADY STATES

Phase of Forced Vibrations. The phase of the forced harmonic oscillator is $pt - \phi$, where p is the angular frequency of the impressed force and ϕ is given by equation (43). Relation (44) shows that the forced vibrations lag behind the impressed force by an angle ϕ, which depends upon both the frequency of the impressed force $(p/2\pi)$ and the damping constant b. Relation (41) shows that $\sin\phi$ is always positive, hence ϕ must lie between 0 and π for all values of p and b. Relation (40) shows that

$\cos\phi$ is positive, ϕ lies between 0 and $\pi/2$ for $p < \omega_0$
$\cos\phi$ is negative, ϕ lies between $\pi/2$ and π for $p > \omega_0$
$\cos\phi = 0$, $\phi = \pi/2$ for $p = \omega_0$.

Thus we see that forced oscillations lag behind the impressed force by a quarter period when the frequency of the impressed force coincides with the natural frequency of the oscillator. Equation (43) also shows that

If $p << \omega_0$, $\tan\phi$ is negligible and $\phi = 0$

$p >> \omega_0$, $\tan\phi$ is negligible but a negative quantity and $\phi = \pi$.

Thus we conclude that in the presence of damping, as the forcing frequency $(p/2\pi)$ is increased from 0 to the natural frequency $\omega_0/2\pi$, the phase lag ϕ increases from 0 to $\pi/2$. If p is further increased beyond ω_0, ϕ increases from $\pi/2$ to π.

Equation (43) also shows that

(*i*) If the damping is very small say zero, ϕ is zero for $0 \le p < \omega_0$ and is π for $p > \omega_0$. ϕ jumps suddenly from 0 to π at $p = \omega_0$.

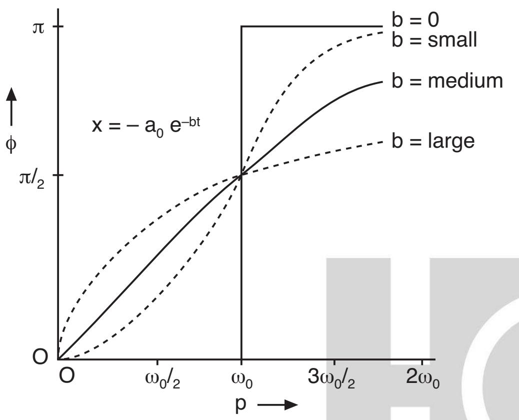

Fig. 1.10. Variation of phase angle ϕ with damping.

(*ii*) For non-zero but small damping, phase lag ϕ is very nearly to zero if $p < \omega_0$. It is very nearly to π if $p > \omega_0$. $\phi = \pi/2$ for $p = \omega_0$. Thus ϕ increases form 0 to $\pi/2$ as p increases from 0 to ω_0 and then increases from $\pi/2$ to π as p further increases beyond ω_0.

(*iii*) As the damping increases, the phase increases from 0 to π, but the rate of increase is slow.

The variation of phase with the frequency of the impressed force for different values of damping b is shown is Fig. 1.10. The curves indicate that the smaller is the value of b, the greater is the rate of change of phase. All these curves pass through $\phi = \pi/2$ at $p = \omega_0$, irrespective of the value of b.

1.9. RESONANCE

Amplitude Resonance. The amplitude of the forced oscillations is given by

$$A = f_0/\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}.$$

It is maximum, when the denominator of the right hand side of the above equation is minimum, *i.e.*, when

$$\frac{d}{dp}[(\omega_0^2 - p^2)^2 + 4b^2p^2] = 0 \quad \text{and} \quad \frac{d^2}{dp^2}[(\omega_0^2 - p^2)^2 + 4b^2p^2] = +\text{ve}.$$

or $$-4p\,(\omega_0^2 - p^2) + 8b^2p = 0,$$

$\therefore$ Either $$p = 0 \quad \text{or} \quad p = \sqrt{\omega_0^2 - 2b^2}.$$

On further differentiating the denominator term, we get

$$\frac{d}{dp}\left[-4p\,(\omega_0^2 - p^2) + 8b^2p\right] = -4\,(\omega_0^2 - p^2) + 8p^2 + 8b^2 = 4\,(3p^2 - \omega_0^2 + 2b^2)$$

$$= 4\,[3\omega_0^2 - 6b^2 - \omega_0^2 + 2b^2] = 8\,(\omega_0^2 - 2b^2).$$

This is positive, when $\omega_0^2 > 2b^2$.

Thus we see that the amplitude is maximum when $\omega_0^2 > 2b^2$ and $p = \sqrt{\omega_0^2 - 2b^2}$. The amplitude is minimum (f_0/ω_0^2), when $p = 0$. This is the static displacement of the body due to the force F_0. The amplitude again approaches to the minimum when p increases indefinitely above ω_0. The *phenomenon of production of forced oscillations of maximum* amplitude is called the *amplitude resonance* and the frequency of the impressed force at which resonance occurs is called the *resonant frequency*. Thus the resonant frequency is given by

$$f = \nu_R = p_R/2\pi = \sqrt{\omega_0^2 - 2b^2}/2\pi. \quad \ldots(46)$$

This is always less than the natural frequency ($\omega_0/2\pi$) of the body.

Substituting this value of p in equation (42), we get the expression for the maximum amplitude as

$$A_{\max} = f_0/2b\sqrt{\omega_0^2 - b^2}. \quad \ldots(47)$$

This shows that the smaller the value of damping b, the greater will be the value of maximum amplitude. For $2b^2 << \omega_0^2$, we have $p \simeq \omega_0$ and the maximum amplitude

$$A_{\max} = f_0/2b\omega_0. \quad \ldots(48)$$

Thus for small damping, the amplitude of forced oscillations becomes maximum, when the frequency of the impressed force equals the natural frequency of the body, *i.e.*, $p = \omega_0$. Obviously, for no damping (*i.e.*, for $b = 0$), the amplitude becomes infinite. *It is an ideal condition, never occurs in practice as damping can never be zero.*

The graph between A and p for different values of damping, Fig. 1.11, shows that for $p = \omega_0$ or $b = 0$, the curve becomes asymptotic to the amplitude axis.

The peak value of A (*i.e.*, $A_{\max}$) is different for different values of damping. For $b = 0$, $A_{\max} = \infty$. This value decreases as we increase the damping b. When the damping is too large, so that $2b^2 > \omega_0^2$, p becomes imaginary. In such a case the amplitude decreases continuously as p increases, without taking place any resonance.

The value of p corresponding to the peak of any curve gives the resonance frequency for that curve, *i.e.*, for that damping b. The resonant frequency is closer to the natural frequency ω_0 of the body for the smaller value of b, *i.e.*, $p_R \to \omega_0$ as $b \to 0$. The fall in the curves on either side of $p = \omega_0$ is steeper in the case of smaller damping than in the case of heavier one.

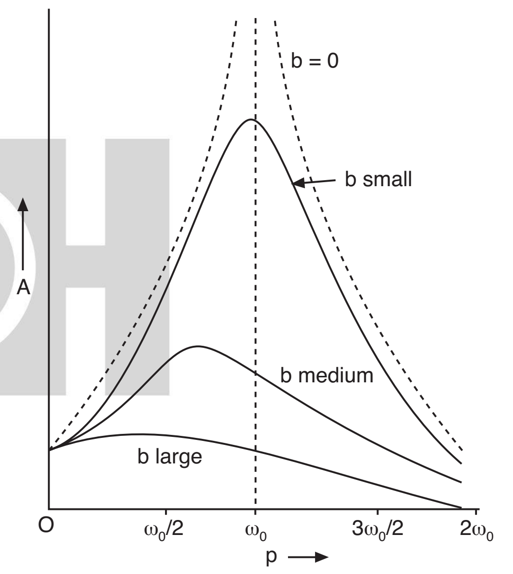

Fig. 1.11. Variation of amplitude with damping.

Velocity Resonance. The velocity of the forced harmonic oscillator at any instant t may be obtained by differentiating equation (44) with respect to t, as

$$v = \frac{dx}{dt} = \frac{pf_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \cos(pt - \phi). \quad \ldots(49)$$

For the given values of p and b, the velocity is *maximum*, when $\cos(pt - \phi)$ is maximum (*i.e.*, equal to one) or $pt - \phi = 0$. It is the condition for zero displacement. The maximum velocity at the position of zero displacement of the body is called the *velocity amplitude* and is given by

$$v_0 = pf_0/\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}.$$

For a given value of b, $v_0 = 0$, when $p = 0$ and is maximum, when

$$\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}/p = \sqrt{(\omega_0^2/p - p)^2 + 4b^2}.$$

is minimum or $p = \omega_0$.

$$\therefore \quad (v_0)_{max} = f_0/2b. \quad ...(50)$$

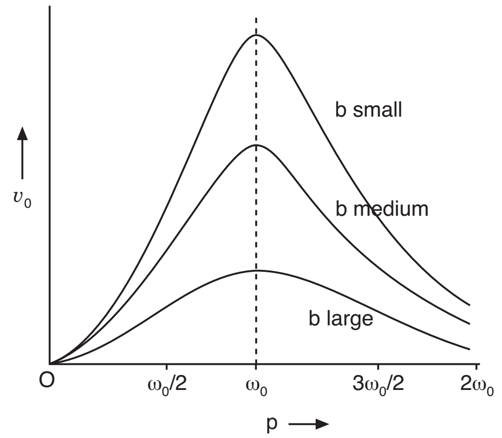

Fig. 1.12. Velocity resonance curve.

Thus we see that regardless of the damping value, a forced harmonic oscillator attains its maximum velocity amplitude, when the frequency of the impressed force is equal to the natural frequency of the body. *The phenomenon of production of forced oscillations of maximum velocity amplitude is called the velocity resonance.*

The equation (50) shows that the maximum velocity amplitude is inversely proportional to the damping constant b. The variation of velocity amplitude v_0 with p for different values of b is shown in Fig. 1.12. The curves show that

(*i*) Smaller the value of b, the greater will be the value of maximum velocity amplitude.

(*ii*) The curves are nearly symmetrical about the vertical line $p = \omega_0$.

The relation for velocity, equation (49), shows that the velocity of the impressed force leads the displacement by $\pi/2$. As the displacement at resonance lags behind (in phase) the impressed force by $\pi/2$, therefore *at resonance the velocity must be in phase with the impressed force.*

1.10. SHARPNESS OF RESONANCE

We have seen above that the amplitude of forced oscillations decreases as the damping increases. Intensity of the oscillations is proportional to the square of the amplitude of the oscillations. The measure of intensity is called the response of forced oscillations. Hence the *response of a body or system to forced oscillations decreases as the damping increases.*

If R be the response of a system to forced oscillations and R_{max} that at resonance, then we have

$$\frac{R_{max}}{R} = \left(\frac{A_{max}}{A}\right)^2 = \frac{(\omega_0^2 - p^2)^2 + 4b^2p^2}{4b^2(\omega_0^2 - b^2)}.$$

Let $\omega/2\pi$ be the resonant frequency, then from equation

$$\omega^2 = \omega_0^2 - 2b^2 \quad \text{or} \quad \omega_0^2 = \omega^2 + 2b^2$$

we have

$$\frac{R_{max}}{R} = \frac{(\omega^2 + 2b^2 - p^2)^2 + 4b^2p^2}{2b^2[\omega^2 + b^2]}$$

or

$$\frac{R_{max} - R}{R} = \frac{p^4 - 2p^2\omega^2 + \omega^4}{4b^2(\omega^2 + b^2)} = \frac{(p^2 \sim \omega^2)^2}{4b^2(\omega^2 + b^2)}.$$

Thus for a slight departure from the resonance condition, $p \sim \omega = \Delta$ and $p + \omega \simeq 2\omega$. Hence the above equation becomes

$$\frac{R_{max} - R}{R} = \frac{\omega^2\Delta^2}{b^2(\omega^2 + b^2)} = \frac{\Delta^2}{b^2(1 + b^2/\omega^2)}.$$

For $b^2 << \omega^2$, this reduces to

$$(R_{max} - R)/R = \Delta^2/b^2. \quad ...(51)$$

Thus we see that quantity Δ, which measures the degree of mistuning, is always positive.

Relation (51) represents the want of response per unit response of the system to forced oscillations and is a measure of *sharpness of response.* It is large when the damping (b) is small and is small for large damping. In other words *when the damping is small the response falls off very rapidly on either sides of resonance, i.e., resonance is sharp. The response falls off very slowly on either sides of resonance, i.e., resonance is flat, for the large damping.*

Thus the *sharpness of resonance is a measure of the rate of fall of amplitude or the response from its maximum value at the resonance frequency, on either side of it. The sharper the fall, the larger the sharpness of resonance. The sharpness of resonance may be defined by equation (51)* which shows that the sharpness of resonance is inversely proportional to the square of the damping constant b.

Half Width of Resonance Curve. We have seen that the frequency amplitude curve of a forced harmonic oscillator is nearly symmetric about the resonance frequency ($p_R = \omega$) for small damping. The frequency of the oscillator at which its amplitude reduces to the half of its maximum value ($\frac{1}{2} A_{\max}$) is denoted by p_H. It is the measure of the width of resonance curve. The change in frequency of the forced harmonic oscillator corresponding to the change in amplitude from its maximum to half this maximum is called the *half width of the resonance curve.*

$\therefore$ Half width of the resonance curve = $\Delta p = | p_H - p_R |$.

Thus from the definition of half width of the resonance curve

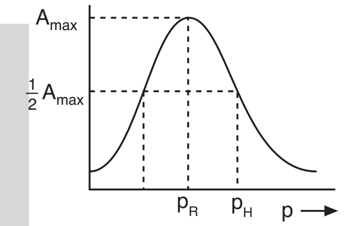

Fig. 1.13. Half width of resonance curve.

$$\tfrac{1}{2} A_{max} = A_{p = p_H}$$

or
$$\frac{f_0}{4b\sqrt{\omega_0^2 - b^2}} = \frac{f_0}{\sqrt{(\omega_0^2 - p_H)^2 + 4b^2 p_H^2}}$$

or
$$16b^2 (\omega_0^2 - b^2) = (\omega_0^2 - p_H)^2 + 4b^2 p_H^2.$$

Since the resonance frequency is given by the relation $p_R^2 = \omega_0^2 - 2b^2$, therefore the above relation reduces to

$$(p_R^2 - p_H^2)^2 \simeq 3\,(4b^2 p_R^2)$$

or
$$p_H^2 = p_R^2 \mp 2\sqrt{3}\, b p_R = p_R^2 (1 \mp 2\sqrt{3}\, b/p_R)$$

$\therefore$
$$p_H = p_R (1 \mp 2\sqrt{3}\, b/p_R)^{1/2} \simeq p_R (1 \mp \sqrt{3}\, b/p_R)$$
$$\simeq p_R \mp \sqrt{3} b. \qquad \ldots(52)$$

Thus for low damping, the half width of the resonance curve

$$\Delta p \simeq p_H \sim p_R \simeq \sqrt{3} b. \qquad \ldots(53)$$

Thus we see that the damping constant b can be obtained from the resonance curve.

1.11. POWER DISSIPATION AND QUALITY FACTOR

Energy of Oscillation. The total energy of the forced harmonic oscillator is equal to the maximum value of its kinetic energy. It is thus given by

$$E = (\text{K.E.})_{\max} = [\tfrac{1}{2} mv^2]_{\max} = \tfrac{1}{2}m\, [v_0 \cos (pt - \phi)]^2_{\max}$$
$$= \frac{1}{2} m v_0^2 = \frac{1}{2} m \cdot \frac{p^2 f_0^2}{(\omega_0^2 - p^2)^2 + 4b^2 p^2} = \frac{1}{2} m \, \frac{f_0^2}{(\omega_0^2/p - p)^2 + 4b^2}$$

This is maximum, when $p = \omega_0$ and has the value

$$E_{\max} = m f_0^2 / 8b^2. \qquad \ldots(54)$$

This shows that the energy of the forced harmonic oscillator is maximum at $p = \omega_0$, *i.e.,* at velocity resonance or when the frequency of the impressed force is equal to the natural frequency of the oscillating body. The energy becomes infinite at $p = \omega_0$ when the damping b is zero, which is not practicable.

Power in the Forced Harmonic Oscillator. In forced harmonic oscillations, the energy is dissipated in doing work against damping forces. This energy is supplied by the impressed force, which may be evaluated as follows :

The damping force per unit mass of the oscillating body = $2b\,(dx/dt)$.

Since the power $P = F\,(dx/dt)$, therefore the instantaneous value of power dissipated per unit mass is given by

$$P = 2b\,(dx/dt)^2 = 2b\,[pA\cos(pt - \phi)]^2.$$

Hence the energy dissipated per unit mass during one complete cycle is given by

$$E_{\text{dissipated}} = \int_0^T 2b\,p^2A^2\cos^2(pt-\phi)\,dt = bp^2A^2\int_0^T [1 + \cos 2(pt-\phi)]\,dt$$

$$= bp^2A^2\,T = bp^2A^2\,(2\pi/p) = 2\pi bpA^2.$$

∴ Average power dissipated per unit mass

$$(P_{av})_{\text{dissipated}} = E_{\text{dissipated}}/T = bp^2A^2. \qquad \ldots(55)$$

The instantaneous power supplied per unit mass to the oscillator by the impressed force is given by

$$P_s = (f_0 \sin pt)\,[pA\cos(pt-\phi)].$$

Hence the energy supplied per unit mass during one complete cycle is given by

$$E_{\text{supplied}} = \int_0^T f_0 pA \sin pt \cos(pt-\phi)\,dt$$

$$= \frac{1}{2} f_0 pA \int_0^T [\sin(2pt-\phi) + \sin\phi]\,dt \qquad \ldots(56)$$

$$= \tfrac{1}{2} f_0 pA\,[0 + T\sin\phi] = \tfrac{1}{2} f_0 pA\,T\sin\phi.$$

∴ Average power supplied per unit mass

$$(P_{av})_{\text{supplied}} = E_{\text{supplied}}/T = \tfrac{1}{2} f_0 pA \sin\phi.$$

From equation (41), we know that $f_0 \sin\phi = 2bpA$, therefore

$$(P_{av})_{\text{supplied}} = bp^2A^2. \qquad \ldots(57)$$

Thus we see that the average power dissipated in doing work against damping forces is equal to the average power supplied by the impressed force. In the steady state, amplitude and phase of the forced oscillations adjust themselves to attain this condition.

It is clear from equation (56) that the energy supplied during the part of oscillation consists of two terms, the first term $[\sin(2pt-\phi)]$ fluctuates in magnitude and alternates in sign and having a zero average over a whole period. When this term becomes negative, the energy is received from the oscillator itself. The second term is constant and represents the average power supplied by the impressed force.

Substituting the value of A in equations (55) and (57), we have

$$(P_{av})_{\text{supplied}} = (P_{av})_{\text{dissipated}} = bp^2\frac{f_0^2}{(\omega_0^2 - p^2)^2 + 4b^2p^2} = \frac{f_0^2 b}{(\omega_0^2/p - p)^2 + 4b^2}.$$

This is maximum, when $p = \omega_0$, *i.e.*, at velocity resonance. Substituting $p = \omega_0$, the natural frequency of the oscillator in the above equation, we get

$$(P_{av})_{max} = f_0^2/4b. \quad \ldots(58)$$

Thus we see that the maximum power supplied to the oscillator or dissipated by it varies inversely as the damping constant b. It is zero for undamped oscillator ($b = 0$) except at resonance.

Half Width of Average Power Curve. We know that the average power dissipated or absorbed per unit mass on account of damping is given as

$$P_{av} = f_0^2 b/[(\omega_0^2/p - p)^2 + 4b^2]$$

and $$(P_{av})_{max} = f_0^2/4b. \quad \ldots(59)$$

Similar to the half width of the amplitude resonance curve, the change in frequency of the forced oscillator corresponding to the change in average power from its maximum to half this maximum is called the *half width of average power* v/s frequency curve.

$$\Delta p = | P_R - P_H |$$

$$\frac{1}{2} P_{max} = (P_{av})_H$$

or $$\frac{f_0^2}{8b} = \left[\frac{f_0^2 b}{(\omega_0^2/p - p)^2 + 4b^2}\right]_{p = p_H}$$

or $$(\omega_0^2/p_H - p_H)^2 = 4b^2 \quad \text{or} \quad \omega_0^2 - p_H^2 = \pm 2bp_H.$$

or $$\omega_0 - p_H = \pm \frac{2bp_H}{\omega_0 + p_H} = \pm \frac{2b}{\omega_0/p_H + 1}. \quad \ldots(60)$$

Since ω_0/p_H may be assumed as about unity, hence

Half width of the power versus frequency curve

$$\Delta p = | \omega_0 - p_H | \simeq b. \quad \ldots(61)$$

Since $\omega_0 = p_R$, *i.e.*, the frequency at which average power is maximum is same as that the natural frequency of the harmonic oscillator. Hence Δp the half width of the average power absorbed or dissipated versus the frequency of the impressed force curve is equal to the damping constant b or $1/2\tau$.

Quality factor of the Forced Harmonic Oscillator. We have defined the quality factor Q as 2π times the ratio between the energy stored and the energy dissipated per period/cycle.

$$Q = 2\pi \frac{(E_{av})_{stored}}{(E_{av})_{\text{dissipated per cycle}}}.$$

For a forced harmonic oscillator, the energy at any instant t is given as

$$E(t) = (\text{K.E.})_t + (\text{P.E.})_t = \frac{1}{2} m\, p^2 A^2 \cos^2 (pt - \phi) + \frac{1}{2} m\, \omega_0^2 A^2 \sin^2 (pt - \phi)$$

$$\therefore \text{ Average energy stored} = \frac{1}{T}\int_0^T \left[\frac{1}{2} mp^2 A^2 \cos^2 (pt - \phi) + \frac{1}{2} m\omega_0^2 A^2 \sin^2 (pt - \phi)\right] dt$$

$$= \frac{1}{2} mp^2 A^2 \left(\frac{1}{2}\right) + \frac{1}{2} m\omega_0^2 A^2 \left(\frac{1}{2}\right) = \frac{1}{4} mA^2 (p^2 + \omega_0^2).$$

Since the average power dissipated over a cycle $= mA^2 p^2 b$.

$$\therefore \quad Q = 2\pi \frac{\frac{1}{4} mA^2 (p^2 + \omega_0^2)}{mA^2 p^2 b\, T} = \frac{1}{4} \frac{p^2 + \omega_0^2}{p^2 b/p}$$

$$= \frac{1}{4} (1 + \omega_0^2/p^2) p/b. \quad \ldots(62)$$

Since at resonance $p \simeq \omega_0$, hence we have

$$Q = \frac{1}{4} \times 2 \times p/b = p/2b = p\tau = \omega_0 \tau. \quad \ldots(63)$$

It is same as defined earlier.

Since the half width of the average power absorbed or dissipated versus frequency curve

$$\Delta p = b = 1/2\tau \quad \text{or} \quad \tau = 1/2\Delta p.$$

$\therefore$ $Q = \omega_0\tau = \omega_0/2\Delta p$ = resonance frequency/full width of power frequency curve at half its maximum

We thus see that near resonance ($p \simeq \omega_0$), $Q = \omega_0\tau$. For low damping Q is high. Hence the quality factor Q is a measure of sharpness of resonance.

EXERCISES

Example 1. *A particle of mass 5gm executing S.H.M. has amplitude of 8cm. If it makes 16 vibrations per second, find its maximum velocity and energy at mean position.*

The displacement of a particle in *S.H.M.* is given by

$$x = a \sin(\omega t + \phi),$$

where a is its amplitude, ω, the angular velocity.

Its velocity $v = dx/dt = a\omega \cos(\omega t + \phi)$.

Velocity v will be maximum, when $\cos(\omega t + \phi)$ is maximum, *i.e.,* 1

$\therefore$ $v_{\max} = a\omega = a\ 2\pi n$, where n is the frequency

Given amplitude $a = 8cm = 8 \times 10^{-2} m$, $n = 16$ vibrations / sec and $m = 5gm = 5 \times 10^{-3}\ kg$

$\therefore$ $v_{\max} = 8 \times 10^{-2} \times 2 \times 3.14 \times 16 = 8.04\ m$ /sec.

At mean position, the energy is entirely kinetic, therefore

$$E = \frac{1}{2} m v_{\max}^2 = \frac{1}{2} \times 5 \times 10^{-3} \times (8.04)^2 = 0.1615 \text{ Joule.}$$

Example 2. *The amplitude of harmonic oscillations of a point mass of 10 gm is 5cm and the energy of oscillation is 3.4×10^{-5} Joule. Write the equation of motion if the initial phase is 60°.*

The energy of oscillation $E = \frac{1}{2} ka^2 = \frac{1}{2} m\, \omega^2 a^2$

$\therefore$ $\omega = \sqrt{2E/ma^2}$.

Given that $m = 10gm = 10 \times 10^{-3}\ kg$, $a = 5\ cm = 5 \times 10^{-2}$ and $E = 3.4 \times 10^{-5}$ Joule

$$\therefore \quad \omega = \sqrt{\frac{2 \times 3.4 \times 10^{-5}}{10 \times 10^{-3} \times (5 \times 10^{-2})^2}} = \sqrt{\frac{6.8 \times 10^{-5}}{2.5 \times 10^{-5}}} = 1.65 \text{ rad/sec.}$$

Initial phase $\phi = 60° = 60 \times \pi/180 = \pi/3$ radian.

$\therefore$ Equation of harmonic oscillations is given by

$$x = a \sin(\omega t + \phi)$$
$$= 5 \times 10^{-2} \sin(1.65t + \pi/3)\ m.$$

Example 3. *A particle moves with simple harmonic motion in a straight line. In first τ seconds, after starting from rest it travels a distance a and in the next τ seconds, it travels 2a in the same direction. Find the time period and amplitude of the motion.*

Particle at rest at the extreme side and moved inwards when released. Thus the displacement at $t = 0$, will be its amplitude A. Thus the equation of motion will be

$$x = A \cos \omega t. \qquad \ldots(i)$$

Given the displacement $x = A - a$ at $t = \tau$

and $= A - 3a$ at $t = 2\tau$

$\therefore$ $A - a = A \cos \omega\tau$ and $A - 3a = A \cos \omega(2\tau)$

or $\cos \omega\tau = (A - a)/A$ and $\cos 2\omega t = (A - 3a)/A$

Since $\cos 2\omega\tau = 2\cos^2\omega\tau - 1$,

$$\therefore \quad \frac{A-3a}{A} = 2 \cdot \left(\frac{A-a}{A}\right)^2 - 1 = \frac{2A^2 + 2a^2 - 4aA - A^2}{A^2}$$

or $A^2 - 3aA = A^2 - 4aA + 2a^2$

$\therefore \quad A = 2a.$

$$\therefore \quad \cos \omega\tau = \frac{A-a}{A} = \frac{2a-a}{2a} = \frac{1}{2}.$$

$$\therefore \quad \omega\tau = 60° = \pi/3. \quad \text{or} \quad \frac{2\pi}{T} \cdot \tau = \frac{\pi}{3} \quad \therefore \quad T = 6\tau.$$

Example 4. *A particle of mass 10 gm moves under a potential V(x) given by* $V(x) = 8 \times 10^5 x^2$ *ergs/gm, where x is in centimeters. Deduce the time-displacement relation when the total energy is 0.08 joule.*

When a particle of mass m moves under a potential $V(x)$ it acquires potential energy $U(x) = mV(x)$.

In the present problem, $m = 10$ gm $= 10^{-2}$ kg,

and $V(x) = 8 \times 10^5 x^2$ ergs/gm $= 8 \times 10^5 x^2 \dfrac{\text{joule}}{10^7} \cdot \dfrac{10^3}{\text{kg}} = 80x^2$ joule/kg.

If x is measured in meters, then we have

$$V(x) = 8 \times 10^5\, x^2 \text{ joule/kg.}$$

$$\therefore \quad U(x) = 10^{-2} \times 8 \times 10^5 x^2 = 8 \times 10^3 x^2 \text{ joule.}$$

This energy is maximum and equal to the total energy when the displacement x is maximum (= amplitude A).

$$\therefore \quad \text{Total energy} = (U_x)_{\max} = 8 \times 10^3 A^2 \text{ joule.}$$

But its given value is 0.08 joule, therefore

$$8 \times 10^3 A^2 = 0.08$$

$$\therefore \quad A^2 = 10^{-5} \quad \text{or} \quad A = 10^{-5/2}.$$

The force acting on the particle may be defined as

$$F = -dU/dx$$

$$\therefore \quad m\ddot{x} = -m\, dV/dx \quad \text{or} \quad \ddot{x} = -dV/dx.$$

Substituting the value of dV/dx by differentiating the relation for $V(x)$, we get

$$\ddot{x} = -16 \times 10^5 x \quad \text{or} \quad \ddot{x} + 16 \times 10^5 x = 0.$$

This is the differential equation of motion of the particle representing a simple harmonic motion, given by

$$x = A \sin(\omega t + \theta),$$

where $A = 10^{-5/2}$ and $\omega = (16 \times 10^5)^{1/2} = 400\sqrt{10}$.

Therefore the displacement is given by the equation

$$\therefore \quad x = 10^{-5/2} \sin(400\sqrt{10}\, t + \theta).$$

Example 5. *A particle is executing S.H.M. of period 10 seconds and amplitude 6 cm. If the particle has started at 3 cm from the central position in the positive direction, calculate (i) the epoch of the vibrating particle, (ii) its phase when at a distance of 4 cm from the mean* *position and (iii) the phase difference between any two positions of the vibrating particle 3 seconds apart.*

Let the equation of the motion of the given particle be

$$x = A \sin(\omega t + \theta).$$

The phase of the vibrating particle at any instant t is thus given by

$$\omega t + \theta = \sin^{-1}(x/A).$$

(*i*) As we define the epoch of the vibrating particle as the phase at the initial stage, *i.e.,* at $t = 0$, therefore

$$\text{epoch} = \theta = \sin^{-1}(x/A) = \sin^{-1}(0.03/0.06)$$
$$= \sin^{-1}(1/2) = 30°.$$

(*ii*) The phase of the particle when it is at a distance of 4 cm is thus given by

$$\omega t + \theta = \sin^{-1}(x/A) = \sin^{-1}(0.04/0.06)$$
$$= \sin^{-1}(2/3) = 41.8° \text{ or } 138.2°,$$

according as the vibrating particle is moving away or towards the mean position.

(*iii*) The phases at the instants t_1 and t_2 are given as $\omega t_1 + \theta$ and $\omega t_2 + \theta$ respectively. Therefore the phase difference between two instants $t_2 - t_1$ apart is given as

$$(\omega t_2 + \theta) - (\omega t_1 + \theta) = \omega(t_2 - t_1) = (2\pi/T)(t_2 - t_1)$$
$$= (2 \times 3.14/10) \times 3 = 3\pi/5 \text{ radians.}$$

Example 6. *A body executes linear simple harmonic motion of a period T_1 under one constraining force and of period T_2 under another. Show that its period under a constraining force equal to their sum is given by*

$$T = T_1 T_2 / \sqrt{T_1^2 + T_2^2}.$$

Let the constraining forces be F_1 and F_2 corresponding to the period of motion T_1 and T_2 respectively. If a_1 and a_2 are the accelerations in these two cases respectively, then we have

$$a_1 = F_1/m \text{ and } a_2 = F_2/m,$$

where m is the mass of the body. If a be the acceleration of the body when the constraining force is $F_1 + F_2$, therefore we have

$$a = \frac{F_1 + F_2}{m} = \frac{F_1}{m} + \frac{F_2}{m}$$

or

$$a = a_1 + a_2. \quad \ldots(i)$$

Since the body is executing simple harmonic motion, hence from the definition of simple harmonic motion

$$a = -\omega^2 x,$$

where x is the displacement from the mean position and ω^2 is the constant of proportionality.

As $\omega = 2\pi/T$, therefore

$$a = -(2\pi/T)^2 x = -(4\pi^2/T^2)x.$$

Similarly, we have

$$a_1 = -\frac{4\pi^2}{T_1^2}x \text{ and } a_2 = -\frac{4\pi^2}{T_2^2}x. \quad \ldots(ii)$$

Substituting these values in equation (*i*), we get

$$-\frac{4\pi^2}{T^2}x = -\frac{4\pi^2}{T_1^2}x - \frac{4\pi^2}{T_2^2}x$$

or $$\frac{1}{T^2} = \frac{1}{T_1^2} + \frac{1}{T_2^2} = \frac{T_1^2 + T_2^2}{T_1^2\, T_2^2}$$

or $$T = T_1T_2/\sqrt{T_1^2 + T_2^2}.$$

Example 7. *A vertical U-tube of uniform cross-section contains mercury to a height of 20 cm. If it is depressed on one side and then released, the mercury column oscillates up and down the two sides of the tube. Evaluate the time period of the oscillations.*

Given that the liquid (mercury) is contained in a vertical U-tube of uniform cross-section A. If the liquid is depressed in the left limb, it rises to an equal height in the right limb. On account of gravitation potential due to the different levels in the limbs, the liquid will perform vertical oscillations with a definite time period.

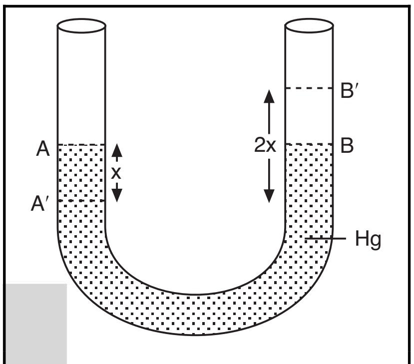

Fig. 1.14. Oscillations in a U-tube.

Let l be the total length of the liquid in the tube and m its mass per unit length. If at any instant the depression of the liquid level in the one limb below its initial position is x (A to A′ in the left limb), the liquid level in the other limb will be at an equal height x above its initial level (B to B' in the right limb). Thus the difference of levels of the liquid in the two limbs is $2x$. The weight of this column of liquid will be $2x.mg$. The force due to this weight will act on the mass ml of the whole liquid. If a be the acceleration of the liquid at this instant, then from Newton's law

$$F = mla = -2x\,mg$$

$$\therefore \quad a = -2xg/l, \quad \text{or} \quad d^2x/dt^2 = -(2g/l)\,x. \qquad ...(i)$$

The above relation shows that the acceleration is proportional to the displacement x and is oppositely directed. Therefore the motion of the liquid is simple harmonic about its initial position. Its time period is given by

$$T = 2\pi\sqrt{l/2g}\,. \qquad ...(ii)$$

Substituting numerical values as, $l = 2 \times 20$ cm $= 0.4$ m and $g = 9.81$ m/s^2, we get

$$T = 2 \times 3.14\sqrt{0.4/2 \times 9.81} = 0.897 \text{ s.}$$

Example 8. *Assuming that a smooth straight tunnel is drilled through the centre of earth. If a ball of mass m and radius r is dropped in it, show that the ball will perform simple harmonic motion. Find its time period.*

The earth is assumed to be a solid and homogenous sphere of centre O, radius R and mass M. Let AB be the tunnel existing between two diametrically opposite points A and B on the surface of the earth. When a ball is dropped at its one end, say A, let P be its position at any instant t, where $OP = r$, the distance of the ball from the center of the earth. We know that, the gravitational force on the ball is that due to sphere of radius r, as its outer shell exerts no force on the ball. If ρ be the density of the earth, then the mass of the earth of radius $r = \frac{4}{3}\pi r^3\rho$, where $\rho = M/\frac{4}{3}\pi R^3$.

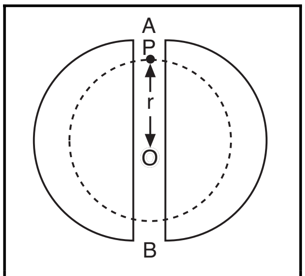

Fig. 1.15. Example 8.

$\therefore$ Force on the ball of mass m is given by

$$F = -G\frac{M'm}{r^2} = -G\cdot\frac{4}{3}\frac{\pi r^3}{r^2}\cdot\frac{M}{\frac{4}{3}\pi R^3}m$$

$$= -GMmr/R^3.$$

The minus sign indicates that the force is attractive and directed toward the center of the earth O. The acceleration at point P is given by

$$a = F/m = -(GM/R^3)\, r = -\omega^2 r,$$

where $\omega^2 = GM/R^3$ = constant. Thus we see that the force or the acceleration is proportional to the displacement from point P and directed toward it. Hence the motion of the ball is simple harmonic. Its periodic time

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{GM/R^3}} = 2\pi \left(\frac{R^3}{GM}\right)^{1/2}$$

$$= 2\pi \left(\frac{R^3}{\frac{4}{3}\pi R^3 \rho G}\right)^{1/2} = 2\pi \left(\frac{3}{4\pi\rho G}\right)^{1/2} = \left(\frac{3\pi}{\rho G}\right)^{1/2}.$$

Example 9. *A hydrogen atom has a mass of 1.68×10^{-27} kg. When attached to a certain massive molecule, it oscillates as a classical oscillator with a frequency of 10^{14} cycles/s and with an amplitude of 10^{-11} m. Calculate the maximum force acting on the hydrogen atom.*

We know that the frequency of a particle executing linear simple harmonic motion is related with its angular velocity as

$$\nu = \omega/2\pi \quad \text{or} \quad \omega = 2\pi\nu = 2 \times 3.14 \times 10^{14}.$$

In a simple harmonic motion, the acceleration of a particle is related with its displacement from its equilibrium position as

$$a = \ddot{x} = -\omega^2 x.$$

Here negative sign shows that the acceleration is towards the equilibrium position (*i.e.,* $x = 0$). At $x = A$ = amplitude, we have the maximum value of acceleration.

maximum acceleration $a = \omega^2 A$

and maximum force $F_{\max} = ma = m\omega^2 A$

$$= (1.68 \times 10^{-27}) \times (2 \times 3.14 \times 10^{14})^2 \times 10^{-11} = 6.63 \times 10^{-9} N.$$

Example 10. *Show that the average value of kinetic energy of a harmonic oscillator is equal to the average value of its potential energy.*

For a harmonic oscillator of mass m, let the equation of motion be

$$x = A \sin(\omega t + \theta).$$

At this displacement, the potential energy

$$= \tfrac{1}{2} m\omega^2 x^2 = \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \theta).$$

For a whole period T, the average potential energy is given by

$$(\text{P.E.})_{av} = \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \theta)\, dt$$

$$= \frac{m\omega^2 A^2}{4T}\int_0^T \left[1 - \cos 2(\omega t + \theta)\right] dt = \frac{m\omega^2 A^2}{4}.$$

On the differentiation of the equation for x with respect to t, we get

velocity $v = \dot{x} = A\omega \cos(\omega t + \theta).$

∴ K.E of the particle at a displacement x

$$= \tfrac{1}{2} m \dot{x}^2 = \tfrac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \theta).$$

For a whole period T, the average kinetic energy is given by

$$(\text{K.E.})_{\text{av}} = \frac{1}{T}\int_0^T \frac{1}{2} mA^2\omega^2 \cos^2(\omega t + \theta)\, dt$$

$$= \frac{m\omega^2 A^2}{4T}\int_0^T \left[1 + \cos 2(\omega t + \theta)\right] dt = \frac{1}{4} m\omega^2 A^2.$$

Thus we see that

$$(\text{K.E.})_{\text{av}} = (\text{P.E.})_{\text{av}} = \tfrac{1}{4} m\omega^2 A^2 = \tfrac{1}{2} E_{\text{max}}.$$

Example 11. *A 10 kg body suspended from the end of a vertical spring of negligible mass stretches it 10 cm. If the damping force at any instant is 80 times its velocity at that instant, then calculate the amplitude, period and frequency of the motion. Given that the body is pulled down initially to 5 cm and then released.*

Consider a coordinate system such that the positive x-axis is downward with origin at the equilibrium. The equation of motion of the spring with a damping force is given by

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx$$

or $$d^2x/dt^2 + (\gamma/m)\, dx/dt + (k/m)\, x = 0.$$

Given that $m = 10$ kg, $\gamma = 80$ and $k = mg/l = 10 \times 9.8/0.1$.

Substituting these values, we have

$$d^2x/dt^2 + 8\, dx/dt + 98x = 0.$$

Its solution will be

$$x = A\, e^{(-4 \pm \sqrt{16-98})\, t} = A\, e^{(-4 \pm 9i)\, t}$$

$$= e^{-4t}(A_1 \cos 9t + A_2 \sin 9t). \quad \ldots(i)$$

Given that initially, *i.e.,* at $t = 0, x = 0.05$ and velocity $dx/dt = 0$, therefore

$$0.05 = e^{-4t} A_1 \text{ or } A_1 = 0.05$$

and $$0 = -4(A_1) + 9A_2 \text{ or } A_2 = (4/9) \times 0.05 = 0.022.$$

Substituting the values of A_1 and A_2 in the equation (i), we get

$$x = e^{-4t}(0.05 \cos 9t + 0.022 \sin 9t)$$

$$= 0.0546\, e^{-4t} \sin(9t + \theta).$$

This shows that the motion is damped oscillatory with amplitude $0.0546\, e^{-4t}$, period $2\pi/9$ and frequency $9/2\pi$.

Example 12. *An under damped harmonic oscillator oscillates with a time period of 10 s. If its first amplitude of 20 cm reduces to 5 cm after 20 oscillations, calculate the (i) damping constant, (ii) relaxation time and (iii) the first amplitude of the oscillator if there was no damping.*

We know that the amplitude of the underdamped harmonic oscillator is given by

$$a = a_0\, e^{-bt}, \quad \ldots(i)$$

where a_0 is the amplitude of the oscillator if there was no damping.

The first amplitude occurs at $t = T/4 = 10/4$ s, hence

$$a_1 = a_0\, e^{-b \times 2.5} = 0.20. \quad \ldots(ii)$$

The amplitude after 20 oscillations occurs after $20T + T/4$ s and is known as 41th amplitude, hence

$$a_{41} = a_0\, e^{-b\,(20T + T/4)} = a_0\, e^{-b\,(202.5)} = 0.05. \quad \ldots(iii)$$

From equations (*ii*) and (*iii*), we get

$$\frac{a_1}{a_{41}} = e^{-b\,(2.5 - 202.5)} = 4$$

or $$200b = \log_e 4.$$

$\therefore$ Damping factor $b = 6.95 \times 10^{-3}\, s^{-1}$.

The relaxation time $\tau = 1/2b = 72$ s.

Amplitude without damping a_0 can be obtained from equation (*ii*) as

$$a_0 e^{-2.5b} = 0.2 \quad \text{or} \quad a_0 = 0.2\, e^{2.5 \times 6.95 \times 10^{-3}}$$

$$\therefore \quad a_0 = 0.2035 \text{ m} = 20.35 \text{ cm}.$$

Thus we see that the damping effect reduces the first amplitude from 20.35 cm to 20 cm.

Example 13. *In an experiment on forced oscillations, the frequency of a sinusoidal driving force is changed while its amplitude is kept constant. It is found that the amplitude of vibrations is 0.01 mm at very low frequency of the driver and goes upto a maximum of 5 mm at driving frequency 200 s^{-1}. Calculate (i) Q of the system, (ii) relaxation time τ and (iii) half width of resonance curve.*

We know that the amplitude of the motion of a forced harmonic oscillator is given by

$$A = f_0/\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}.$$

The amplitude is maximum, when $p^2 = \omega_0^2 - 2b^2$ and is given by

$$A_{max} = f_0/2b\,\sqrt{p^2 + b^2} = f_0/2b\,\sqrt{\omega_0^2 - b^2}.$$

It is clear from the given data that the amplitude increases to 500 times the initial value, *i.e.,* from 0.01 mm to 5 mm. This sharp increase in amplitude is only possible when the damping is too small.

Therefore the amplitude at low frequency, say $p \to 0$,

$$A = f_0/\omega_0^2 = 0.01 \times 10^{-3} \text{ m}. \quad \ldots(i)$$

The maximum amplitude at $p = 2\pi\nu = 2\pi \times 200$,

$$A_{max} = f_0/2bp = f_0/2b\omega_0 = 5 \times 10^{-3} \text{ m}. \quad \ldots(ii)$$

From above equations, we get

$$\omega_0/2b = \omega_0\tau = 500$$

or quality factor $Q = \omega_0\tau = 500$.

Since $\omega_0 = p = 2\pi \times 200$, hence

Relaxation time $\tau = Q/\omega_0 = 500/2\pi \times 200 = 0.4$ s.

The half width of resonance curve (for low damping) is given by

$$\Delta p = \sqrt{3}\, b = \sqrt{3}/2\tau = \sqrt{3}/2 \times 0.4$$
$$= 2.165 \text{ radian/s} = 0.345 \text{ cycles/s}.$$

Example 14. *The amplitude of a damped oscillator decreases to 0.9 times its original magnitude in 5sec. In another 10 sec, it will decrease to α times its original magnitude. Find the value of α.*

We know that in a damped oscillator, the amplitude decreases with time on account of damping, as

$$A = A_o e^{-\lambda t}.$$

Given that at $t = 5$ sec, $A = 0.9\, A_o$

and after another 10 sec, *i.e.*, $t = 15$ sec, $A = \alpha A_o$

Substituting these values, we get

$$0.9\, A_o = A_o e^{-\lambda 5} \quad \text{or} \quad e^{-5\lambda} = 0.9$$

and $$\alpha A_o = Aoe^{-15\lambda} \quad \text{or} \quad e^{-15\lambda} = \alpha$$

From these relations, we get

$$\alpha = (0.9)^3 = 0.729$$

Example 15. *A block of mass 200 gm is attached at the lower end of a spring, suspended vertically from a rigid support. If the force constant k of the spring is 90 Nm*$^{-1}$*. If the arrangement has damping constant b is 50 gm / sec. Find (i) the period of oscillation, (ii) time taken for its amplitude of vibrations to drop to one half of its initial value and (iii) the time taken for its mechanical energy to drop to half of its initial value.*

Restoring force on the block $F = -ky$, where k is the damping constant.

Damping force on the block $F_d = -b\,(dy/dt)$.

From Newton's law of motion

$$m\frac{d^2y}{dt^2} = -ky - b\,(dy/dt)$$

or $$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

This is the equation of a damped harmonic motion. Its displacement at an instant t is given by

$$y = y_o e^{-bt/2m} \sin(\omega' t + \phi),$$

where the amplitude $A = y_o e^{-bt/2m}$ and angular frequency $\omega' = \sqrt{k/m - b^2/4m^2}$.

In our problem, $m = 200\, gm = 0.2\, kg$; $k = 90\, Nm^{-1}$ and $b = 50\, gm/\text{sec} = 0.05 kg / sec$.

$$\therefore \quad k/m = 90/0.2 = 450\, Nm^{-1}kg^{-1}$$

$$b^2/4m^2 = (0.05)^2/\, 4 \times (0.2)^2 = 25/1600\, sec^{-2}.$$

$\therefore$ angular frequency $\omega' = \sqrt{450 - 25/1600} = 21.2\, sec^{-1}$

or time period $T = \dfrac{2\pi}{\omega'} = \dfrac{2 \times 3.14}{21.2} = 0.296 sec.$

Time in which amplitude reduces to its half of initial value y_o

$$\frac{1}{2} y_o e^{-bT'/2m} \quad \text{or} \quad bT'/2m = \log_e 2,$$

$$\therefore \quad T' = (2m/b)\log_e 2 = (2 \times 0.2 / 0.05) \times 2.3026 \times 0.3010$$

$$= 5.54\, sec$$

Mechanical energy varies directly as the square of the amplitude and thus given as

$$E = E_o e^{-bt/m}.$$

Time T'' in which the mechanical energy reduces to half of its initial value is given by

$$\frac{1}{2} E_o = E_o e^{-bT''/m}$$

or $$T'' = (m/b)\log_e 2 = 2.77\, sec.$$

MULTIPLE CHOICE QUESTIONS

1. The displacement of a particle executing simple harmonic motion in one time period is
(*a*) A, (*b*) 2A, (*c*) 4A, (*d*) 0.

2. The average acceleration in one time period of a harmonic oscillator is
(*a*) $A\omega^2$, (*b*) $A\omega^2/2$, (*c*) $A\omega^2/\sqrt{2}$, (*d*) None of these.

3. The potential energy of a harmonic oscillator is maximum when the displacement is
(*a*) 0, (*b*) $\pm a$, (*c*) $\pm a/2$, (*d*) $\pm a/\sqrt{2}$.

4. A simple pendulum, suspended from the roof of a car, has a time period T, when the car is at rest. If the car accelerates with an acceleration a, then the time period of the pendulum will be
(*a*) T, (*b*) more than T, (*c*) less than T, (*d*) None of these.

5. A bob of mass m is attached as shown in Fig. 1.16. When displaced, the bob will oscillate with a time period given by
(*a*) $2\pi\sqrt{l/g}$, (*b*) $2\pi\sqrt{m/k}$,
(*c*) $2\pi\,[l/g + m/k]^{1/2}$, (*d*) $2\pi\,[g/l + k/m]^{-1/2}$.

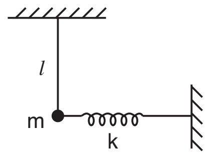

Fig. 1.16. M.P.Q. 5

6. In a harmonic oscillator with amplitude A, the energy is half kinetic and half potential at a distance x from the equilibrium position, where x is
(*a*) $A/4$, (*b*) $A/2$,
(*c*) $3A/4$, (*d*) $(A/2)^{3/2}$.

7. Two bodies A and B of equal mass are suspended by masselss springs of spring constants k_1 and k_2 respectively. The maximum velocities of the bodies are the same when they are oscillating vertically. The ratio of their amplitudes will be
(*a*) k_1/k_2, (*b*) $(k_1/k_2)^{1/2}$, (*c*) k_2/k_1, (*d*) $(k_2/k_1)^2$.

8. A particle is fastened at the end of a string and is whirled in a vertical circle with the other end of the string being fixed. The motion of the particle is
(*a*) simple harmonic, (*b*) periodic, (*c*) oscillatory, (*d*) None of these.

9. The motion of a torsional pendulum is
(*a*) simple harmonic, (*b*) periodic, (*c*) oscillatory, (*d*) None of these.

10. A tunnel is dug along a diameter of the earth. A particle is dropped from a point, a distance 10 km directly above the tunnel. The motion of the particle as seen from the earth is
(*a*) simple harmonic, (*b*) on a straight line, (*c*) parabolic, (*d*) periodic.

11. Which of the following will change the time period as they are taken to moon ?
(*a*) a simple pendulum, (*b*) a physical pendulum,
(*c*) a torsional pendulum, (*d*) a spring mass system.

12. The amplitudes of the periodic motion generally die out with time due to
(*a*) its large amplitude (*b*) small amplitude, (*c*) damping forces, (*e*) resistive forces.

13. At low velocities, the damping force varies as
(*a*) velocity v, (*b*) $v^{1/2}$, (*c*) v^2, (*d*) None of these.

14. The amplitude of damped harmonic oscillator
(*a*) remains constant, (*b*) decreases exponentially,
(*c*) decreases harmonically, (*d*) None of these.

15. If the amplitude of a damped harmonic oscillator decreases suddenly, the oscillator is said to be
(*a*) underdamped, (*b*) overdamped, (*c*) critically damped, (*d*) None of these.

16. The meaning of high quality factor of an oscillator is
(*a*) zero damping, (*b*) small damping, (*c*) high damping, (*d*) infinite damping.

17. The oscillations are underdamped if
(*a*) $b > \omega_0$, (*b*) $b < \omega_0$, (*c*) $b = \omega_0$, (*d*) $b > \omega_0^2$.

18. Which of the following is not correct about forced oscillations ?
(*a*) The transient solution is effective at the initial stage,
(*b*) The free (damped) oscillations die out when $bt >> 1$.
(*c*) The frequency of oscillations is same as that of the impressed force,
(*d*) The forced harmonic oscillator oscillates with its natural frequency.

19. Forced vibrations lag behind the impressed force by an angle ϕ, where
(*a*) ϕ lies between 0 and $\pi/2$ for $p < \omega_0$,
(*b*) ϕ lies between 0 and $\pi/2$ for $p > \omega_0$,
(*c*) ϕ lies between $\pi/2$ and π for $p = \omega_0$,
(*d*) $\phi = \pi/2$ for $p < \omega_0$,

20. At resonance in forced vibrations,
(*a*) The resonant frequency is generally less than the natural frequency due to damping,
(*b*) The resonant frequency is always equal to the natural frequency,
(*c*) The resonant frequency increases as damping decreases,
(*d*) At the resonance frequency amplitude is infinite for damping zero.

21. Total energy of a simple harmonic motion is E. What will be the kinetic energy of the particle when displacement is half of the amplitude ?
(*a*) $E/2$ (*b*) $E/4$ (*c*) $3E/4$ (*d*) E

22. What will be the speed of the pendulum at a point which is at the half way on the left in terms of time period ?
(*a*) $\pi a/T$ (*b*) $2\pi a/T$ (*c*) $\sqrt{3}\,\pi a/T$ (*d*) $\sqrt{2}\,\pi a\,/T$

23. In damped oscillations, the amplitude of oscillations is reduced to one third of its initial value of 9 *cm* at the end of 100 oscillations. What will be its amplitude of oscillation in *cm* when it completes 200 oscillations.
(*a*) 1 *cm* (*b*) 2 *cm* (*c*) 2.5 *cm* (*d*) 1.5 *cm*

ANSWERS: 1. *d*, 2. *d*, 3. *b*, 4. *c*, 5. *d*, 6. *d*, 7. *d*, 8. *b*, 9. *b*, 10. *b, d*, 11. *a, b*, 12. *c, d*, 13. *a*, 14. *b*, 15. *b*, 16. *b*, 17. *b*, 18. *d*, 19. *a*, 20. *a, c*, 21. *c*, 22. *c*, 23. *a*.

SHORT ANSWER QUESTIONS

1. Define the terms : (*i*) amplitude, (*ii*) frequency, and (*iii*) angular frequency in case of the oscillations of a harmonic oscillator.
2. Why do you call periodic motion a harmonic motion ?
3. What is restoring force ? Is it zero on a particle-executing SHM and at the extreme positions ?
4. Define harmonic oscillator.
5. A glider is attached to a fixed ideal spring, which oscillates on a horizontal frictionless air track. If a coin is placed on its top and oscillates with it. At what points the friction force on the coin greatest/least ?
6. What should you do to the length of the string of a simple pendulum to (*a*) double its period, (*b*) double its frequency and (*c*) double its angular frequency ?
7. What would be the force constant of each half, if a uniform spring is cut in half ? How could the frequency differ ?
8. What is damping force ? How does it depend on velocity at low values ?
9. How does the velocity of a body moving under damping force change with time ?
10. Distinguish : overdamped, underdamped and critically damped oscillations.
11. How does the amplitude of the damped oscillator vary with time.
12. Define : (*a*) Power dissipation, and (*b*) quality factor of a damped oscillator.
13. What are the main differences between the dead beat and ballistic galvanometers.
14. What are the essential features of a ballistic galvanometer ?
15. What are forced harmonic oscillators ?
16. What do you mean by amplitude resonance ? Give the conditions ?
17. Define sharpness of resonance. Explain with graph how the response changes with the frequency and damping ?
18. What are the essential differences between the amplitude resonance and velocity resonance ?
19. How is the power dissipated in the forced harmonic oscillator ?
20. How does the average dissipated power vary with the damping in the forced harmonic oscillator ?
21. What do you mean by the quality factor of the forced harmonic oscillator ?

PROBLEMS

1. Find the time period and angular frequency of an ultrasonic transducer which oscillates at a frequency of 6.7 MHz. [0.15 μs, 4.2×10^7 rad/s]

2. A test tube of weight 10 gm and of external diameter one cm, is floated vertically in water by placing 12 gm of mercury at the bottom of the tube. The tube is depressed a small amount and then released. Find the time of oscillation. [0.531 *s*]

3. A spring is mounted horizontally with its left end held stationary and the right end pulled by a spring balance. If a force of 6*N* causes a displacement of 0.02m. If the spring balance is replaced by a 0.5 kg glider. If it is pulled a distance of 0.03 m along a frictionless air track and released. Find the angular frequency and time period of the oscillations produced. [24.5 rad/s, 0.256 s]

4. A block of mass 600 gm is fastened to a spring with spring constant 60 N/m. The block is pulled a distance x = 10 cm from its equilibrium position at x = 0 on a frictionless surface and released from rest at t = 0.